On second order sufficient optimality conditions for quasilinear elliptic boundary control problems

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Outline

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Problem setting

Consider the following optimal control problem

$$(P) \quad \begin{cases} \min J(u) = \int_\Omega \mathbf{L}(x, y_u(x))\, dx + \int_\Gamma \mathbf{l}(s, y_u(s), u(s))\, d\sigma(x)\,, \\ \text{s.t.} \qquad u_a(s) \le u(s) \le u_b(s) \text{ for a.e. } s \in \Gamma \qquad (u_a, u_b \in L^\infty(\Gamma))\,, \end{cases}$$

where y_u is the solution of the quasilinear elliptic equation

$$\begin{cases} -\operatorname{div}\left[\mathbf{a}(x, y(x))\nabla y(x)\right] + \mathbf{f}(x, y(x)) = 0 & \text{in } \Omega\,, \\ \mathbf{a}(x, y(x))\nabla y(x) \cdot \vec{n}(x) = u(x) & \text{on } \Gamma\,. \end{cases} \tag{1}$$

- Ω is an open convex bounded polygonal set of $\mathbb{R}^2$ with boundary Γ.

Which assumptions on **a** and **f** yield the well-posedness of the state equation?

Outline

Study of the quasilinear equation

'Basic' Assumptions:

1. $\mathbf{a}(\cdot,0) \in L^\infty(\Omega)$, $\mathbf{a}$ is locally Lipschitz continuous in y and

$$\exists \alpha_a > 0 \text{ such that } \mathbf{a}(x,y) \geq \alpha_a \text{ for a.e. } x \in \Omega \text{ and all } y \in \mathbb{R}\,.$$

2. (i) $\mathbf{f}(x,\cdot)$ is monotone non-decreasing for a.a. $x \in \Omega$, for every $M > 0$ there exists $\psi_M \in L^q(\Omega)$ ($q \geq \frac{2p}{p+2}$, $p > 2$) such that $\mathbf{f}(x,y) \leq \psi_M(x)$ for a.e. $x \in \bar{\Omega}$ and all $|y| \leq M$.
   (ii) $\exists \alpha_f > 0$ and $E \subset \Omega$, with $|E| > 0$, such that $\dfrac{\partial \mathbf{f}}{\partial y}(x,y) \geq \alpha_f$ for all $(x,y) \in E \times \mathbb{R}$.

- Main difficulty: The state equation is non-monotone.

Example

$$\begin{cases} -\operatorname{div}\left[(\phi_0(x) + y^2)\nabla y\right] + \exp(y) = 0 & \text{in } \Omega\,, \\ (\phi_0(x) + y^2)\nabla y \cdot \vec{n} = u & \text{on } \Gamma\,, \quad (\phi_0 \in C(\bar{\Omega}),\ \phi_0 \geq \alpha > 0)\,. \end{cases}$$

Existence and uniqueness of a solution to (1)

Theorem

- *For any* $u \in L^s(\Gamma)$, $s > 1$, *the state equation (1) has a unique solution* $y_u \in H^1(\Omega) \cap C^\mu(\bar{\Omega})$ *for some* $\mu \in (0,1)$ *independent of* u.
- *If* $\mathbf{a}$ *is continuous on* $\bar{\Omega} \times \mathbb{R}$, *then there exists*

$$p_0 \geq \frac{6}{3 - \sqrt{5}} \approx 7.854$$

such that for any $u \in L^{p/2}(\Gamma)$, $p \in (2, p_0]$, *the solution* $y_u \in W^{1,p}(\Omega)$.

▲ Reduction from the variable coefficient case to the constant coefficient case (Method by Dauge [1]).

- If the Lipschitz property of $\mathbf{a}$ w.r.t. y fails the uniqueness of a solution to (1) is not guaranteed.

M. Dauge, *Neumann and mixed problems on curvilinear polyhedra*. Integral Equations Oper. Theory, 15, No.2:227-261, 1992.

Differentiability of the control-to-state mapping

Linearization of the state equation around a solution y of (1) yields

$$\begin{cases} -\operatorname{div}\left[\mathbf{a}(x,y)\nabla z(x) + \dfrac{\partial \mathbf{a}}{\partial y}(x,y) z \nabla y\right] + \dfrac{\partial \mathbf{f}}{\partial y}(x,y) z = \quad 0 \quad \text{in } \Omega\,, \\ \left[\mathbf{a}(x,y)\nabla z(x) + \dfrac{\partial \mathbf{a}}{\partial y}(x,y) z(x) \nabla y\right]\cdot \vec{n}(x) = v(x) \;\; \text{on } \Gamma\,. \end{cases} \tag{2}$$

- Given $y \in W^{1,p}(\Omega)$ for any $v \in H^{-1/2}(\Gamma)$ the linearized equation (2) has a unique solution $z_v \in H^1(\Omega)$.
- If $\mathbf{a}$ is continuous on $\bar{\Omega} \times \mathbb{R}$, then there exists $p_0 \geq \frac{6}{3-\sqrt{5}}$ such that for $p \in (2, p_0]$ the control-to-state mapping

$$G : L^{p/2}(\Gamma) \to W^{1,p}(\Omega)\,,\; G(u) = y_u\,, \quad \text{is of class } C^1$$

and for any $v \in L^{p/2}(\Gamma)$ the function $z_v = G'(u)v$ is the unique solution in $W^{1,p}(\Omega)$ of (2) at $y = y_u$.

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Existence of a solution for problem (P)

Assume that

1. $\mathbf{a}$ is continuous on $\bar{\Omega} \times \mathbb{R}$,
2. for any $M > 0$, there exist functions $\psi_{l,M} \in L^1(\Gamma)$, $\psi_{L,M} \in L^1(\Omega)$ such that

$$|\mathbf{L}(x, y)| \leq \psi_{L,M}(x) \quad \text{and} \quad |\mathbf{l}(s, y, u)| \leq \psi_{l,M}(s)\,,$$

for a.e. $x \in \Omega$, $s \in \Gamma$ and $|y|, |u| \leq M$.

Theorem (Existence of an optimal control for (P))

If $\mathbf{l}$ is convex w.r.t. u, then the problem (P) has at least one optimal solution $\bar{u}$.

Differentiability of the objective functional

Let the 'standard' assumptions on the second order differentiability of **a**, **f**, **L** and **l** hold.

The functional $J : L^\infty(\Gamma) \to \mathbb{R}$ is of class C^2 and for every $u, v, v_1, v_2 \in L^\infty(\Gamma)$, it holds

$$J'(u)v = \int_\Gamma \left(\frac{\partial \mathbf{l}}{\partial u}(x, y_u, u) + \varphi_u \right) v \, d\sigma(x)$$

$$\begin{aligned} J''(u)v_1 v_2 = \int_\Gamma \Big\{ & \frac{\partial^2 \mathbf{l}}{\partial y^2}(x, y_u, u) z_{v_1} z_{v_2} + \frac{\partial^2 \mathbf{l}}{\partial y \partial u}(x, y_u, u)(z_{v_1} v_2 + z_{v_2} v_1) \\ & + \frac{\partial^2 \mathbf{l}}{\partial u^2}(x, y_u, u) v_1 v_2 \Big\} \, d\sigma(x) + \int_\Omega \left[\frac{\partial^2 \mathbf{L}}{\partial y^2}(x, y_u) - \varphi_u \frac{\partial^2 \mathbf{f}}{\partial y^2}(x, y_u) \right] z_{v_1} z_{v_2} \, dx \\ & - \int_\Omega \nabla \varphi_u \cdot \left[\frac{\partial^2 \mathbf{a}}{\partial y^2}(x, y_u) z_{v_1} z_{v_2} \nabla y_u + \frac{\partial \mathbf{a}}{\partial y}(x, y_u) \left(z_{v_1} \nabla z_{v_2} + z_{v_2} \nabla z_{v_1} \right) \right] dx \, , \end{aligned}$$

where $z_{v_i} = G'(u)v_i$, is the solution of (2) for $y = y_u$ and $v = v_i$, $i = 1, 2, \ldots$

... $\varphi_u \in W^{1,p}(\Omega)$, is the unique solution of the *adjoint equation*

$$\begin{cases} -\operatorname{div}[\mathbf{a}(x,y_u)\nabla\varphi] + \dfrac{\partial \mathbf{a}}{\partial y}(x,y_u)\nabla\varphi\cdot\nabla y_u + \dfrac{\partial \mathbf{f}}{\partial y}(x,y_u)\varphi = \dfrac{\partial \mathbf{L}}{\partial y}(x,y_u) & \text{in } \Omega\,, \\ [\mathbf{a}(x,y_u)\nabla\varphi]\cdot \vec{n}(x) = \dfrac{\partial \mathbf{l}}{\partial y}(x,y_u,u) & \text{on } \Gamma\,. \end{cases}$$

First order necessary optimality conditions

The first order necessary optimality conditions can be deduced by using the inequality $J'(\bar{u})(u - \bar{u}) \geq 0$ and the differentiability of J.

Theorem

If $\bar{u}$ is a local minimum of (P), then there exists $\bar{\varphi} \in W^{1,p}(\Omega)$ such that

$$\begin{cases} -\operatorname{div}[\mathbf{a}(x,\bar{y})\nabla\bar{\varphi}(x)] + \dfrac{\partial \mathbf{a}}{\partial y}(x,\bar{y})\nabla\bar{\varphi}\cdot\nabla\bar{y} + \dfrac{\partial \mathbf{f}}{\partial y}(x,\bar{y})\bar{\varphi} = \dfrac{\partial \mathbf{L}}{\partial y}(x,\bar{y}) & \text{in } \Omega\,, \\ [\mathbf{a}(x,\bar{y})\nabla\bar{\varphi}]\cdot\vec{n}(x) = \dfrac{\partial \mathbf{l}}{\partial y}(x,\bar{y},\bar{u}) & \text{on } \Gamma\,, \end{cases}$$

$$\int_\Gamma \left(\frac{\partial \mathbf{l}}{\partial u}(x,\bar{y},\bar{u}) + \bar{\varphi}(x) \right) (u(x) - \bar{u}(x))\, d\sigma(x) \geq 0 \quad \text{for all} \quad u_a \leq u \leq u_b\,,$$

where $\bar{y}$ is the state associated to $\bar{u}$.

Necessary and sufficient second order optimality conditions

Theorem

If $\bar{u}$ is a local solution for (P), then $J''(\bar{u})v^2 \geq 0$ holds for all $v \in C_{\bar{u}}$, where

$$C_{\bar{u}} := \left\{ h \in L^2(\Gamma) \,\middle|\, h(x) \begin{cases} \geq 0 & \text{if } \bar{u}(x) = u_a(x) \\ \leq 0 & \text{if } \bar{u}(x) = u_b(x) \\ = 0 & \text{if } \frac{\partial \mathrm{l}}{\partial u}(x, \bar{y}, \bar{u}) + \bar{\varphi}(x) \neq 0 \end{cases} \quad \text{for a.e. } x \in \Gamma \right\}.$$

Conversely, if $\bar{u}$ is a feasible control for problem (P) satisfying the first order necessary conditions and

$$J''(\bar{u})v^2 > 0 \qquad \forall v \in C_{\bar{u}} \setminus \{0\},$$

then there exist $\epsilon > 0$ and $\delta > 0$ such that

$$J(\bar{u}) + \frac{\delta}{2}\|u - \bar{u}\|^2_{L^2(\Gamma)} \leq J(u)$$

for every feasible control u for (P), with $\|u - \bar{u}\|_{L^\infty(\Gamma)} \leq \epsilon$.

- The gap between the second order necessary and sufficient optimality conditions is minimal.

Outline

Numerical example

The following specifications satisfy all assumptions made above:

$$\begin{cases} \Omega = (0,\pi)^2\,, & \\ \mathbf{a}(x,y) = 1 + (x_1+x_2)^2 + y^2\,, & \\ \mathbf{f}(x,y) = 2(\sin^2(x_1) + \sin^2(x_2))y + g(x) & (g \in L^q(\Omega))\,, \\ \mathbf{L}(x,y) = \dfrac{1}{2}(y - y_\Omega(x))^2 & (y_\Omega \in L^q(\Omega))\,, \\ \mathbf{l}(x,y,u) = \dfrac{\lambda}{2}u^2 + \eta(x)u & (\eta \in L^2(\Gamma), \lambda \geq 0)\,. \end{cases}$$

For a particular choise of g, y_Ω, η and λ, the functions

$$\bar{y}(x) = \bar{y}(x_1, x_2) = \sin(x_1)\sin(x_2)\,,\ \bar{\varphi}(x) \equiv 1 \quad \text{and} \quad \bar{u}(x) = \mathrm{Proj}_{[-20,-2]}\{e(x)\}\,,$$
$$e(x) = e(x_1, x_2) = -(1 + (x_1 + x_2)^2)(\sin(x_1) + \sin(x_2))\,,$$

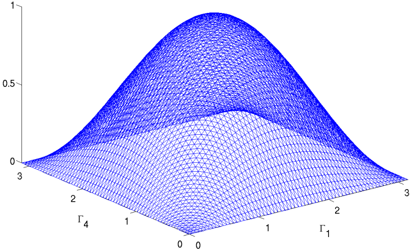
satisfy the first order necessary optimality conditions for the problem

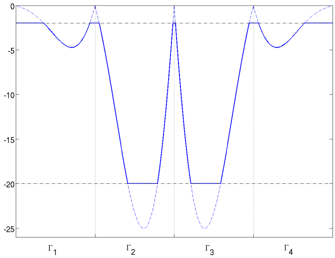
$$(P) \begin{cases} \min J(u) = \displaystyle\int_\Omega \mathbf{L}(x,y)\,dx + \int_\Gamma \mathbf{l}(x,y,u)\,d\sigma(x) \\ \text{s.t.} \quad u \in \{v \in L^\infty(\Gamma) \mid -20 \le v(s) \le -2 \ \text{a.e. } s \in \Gamma\} \\ \text{and} \begin{cases} -\operatorname{div}[\mathbf{a}(x,y)\nabla y] = -\mathbf{f}(x,y) & \text{in } \Omega\,, \\ \mathbf{a}(x,y)\nabla y \cdot \vec{n}(x) = u(x) + \min\{0, e(x)+20\} - \max\{0, e(x)+2\} & \text{on } \Gamma\,. \end{cases} \end{cases}$$

Numerical example

The second order sufficient condition holds for arbitrary non-zero $v \in L^2(\Gamma)$ and $z_v \in H^1(\Omega)$ given by (2):

$$J''(\bar{u})v^2 = \int_\Gamma \left\{ \frac{\partial^2 \mathbf{l}}{\partial y^2}(x,\bar{y},\bar{u})z_v^2 + 2\frac{\partial^2 \mathbf{l}}{\partial u \partial y}(x,\bar{y},\bar{u})vz_v + \frac{\partial^2 \mathbf{l}}{\partial u^2}(x,\bar{y},\bar{u})v^2 \right\} d\sigma(x)$$
$$+ \int_\Omega \left\{ \left[\frac{\partial^2 \mathbf{L}}{\partial y^2}(x,\bar{y}) - \bar{\varphi}\frac{\partial^2 \mathbf{f}}{\partial y^2}(x,\bar{y}) \right] z_v^2 - \nabla\bar{\varphi} \cdot \left[\frac{\partial^2 \mathbf{a}}{\partial y^2}(x,\bar{y})z_v^2 \nabla\bar{y} + 2\frac{\partial \mathbf{a}}{\partial y}(x,\bar{y})z_v \nabla z_v \right] \right\} dx$$
$$= \int_\Gamma v^2 \, d\sigma(x) + \int_\Omega z_v^2 \, dx > 0 \, .$$





Thank you for your attention!