Bang bang control of elliptic equations

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Model problem

$$\min_{u \in U_{ad}} J(u) = \frac{1}{2} \int_\Omega |y - y_0|^2$$
$$\text{subject to } y = \mathcal{G}(u).$$

Here,

$$U_{ad} := \{v \in L^2(\Omega); a \leq u \leq b\} \subseteq L^2(\Omega)$$

with $a < b$ constants, and $y = \mathcal{G}(Bu)$ iff

$$-\Delta y = u \text{ in } \Omega, \text{ and } y = 0 \text{ on } \partial\Omega.$$

More general elliptic operators may be considered, and also control operators which map abstract controls to feasible right-hand sides of the elliptic equation.

Existence and uniqueness, optimality conditions

The optimal control problems admits a unique solution.

The function $u \in U_{ad}$ is a solution of the optimal control problem iff there exists an adjoint state p such that $y = \mathcal{G}(u)$, $p = \mathcal{G}(y - y_0)$ and

$$(p, v - u) \geq 0 \text{ for all } v \in U_{ad}.$$

There holds

$$u(x) \quad \begin{cases} = a, & p(x) > 0, \\ \in [a, b], & p(x) = 0, \\ = b, & p(x) < 0. \end{cases}$$

Strict complementarity requirement for the solution u:

$$\exists C > 0 \forall \epsilon > 0 : \mathcal{L}(\{x \in \bar{\Omega}; |p(x)| \leq \epsilon\}) \leq C\epsilon$$

Variational discretization

Discrete optimal control problem:

$$\min_{u \in U_{ad}} J_h(u) := \frac{1}{2}\int_\Omega |y_h - y_0|^2$$
$$\text{subject to } \; y_h = \mathcal{G}_h(u).$$

Here, $\mathcal{G}_h(u)$ denotes the piecewise linear and continuous finite element approximation to $y(u)$, i.e.

$$a(y_h, v_h) := (\nabla y_h, \nabla v_h) = (u, v_h) \text{ for all } v_h \in X_h,$$

where on a given, quasi-uniform triangulation $\mathcal{T}_h$

$$X_h := \{w \in C^0(\bar{\Omega}); w_{|\partial\Omega} = 0, w_{|T} \text{ linear for all } T \in \mathcal{T}_h\}.$$

This problem is still $\infty-$dimensional.

Ritz projection $R_h : H_0^1(\Omega) \to X_h$,

$$a(R_h w, v_h) = a(w, v_h) \text{ for all } v_h \in X_h$$

Existence and uniqueness, optimality conditions for discrete problem

The variational-discrete optimal control problems admits a unique solution.

The function $u_h \in U_{ad}$ is a solution of the optimal control problem iff there exists an adjoint state p_h such that $y_h = \mathcal{G}_h(u_h)$, $p_h = \mathcal{G}_h(y_h - y_0)$ and

$$(p_h, v - u_h) \geq 0 \text{ for all } v \in U_{ad}.$$

There holds

$$u_h(x) \quad \begin{cases} = a, & p_h(x) > 0, \\ \in [a, b], & p_h(x) = 0, \\ = b, & p_h(x) < 0. \end{cases}$$

Error estimate

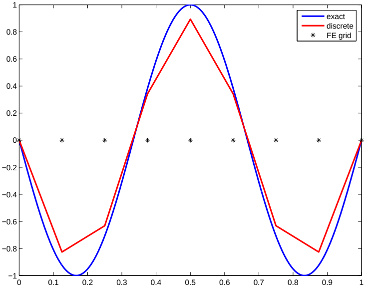
Let u, u_h denote the unique solutions of the optimal control problems with corresponding states $y = \mathcal{G}(u)$ and $y_h = \mathcal{G}_h(u_h)$, resp. Then

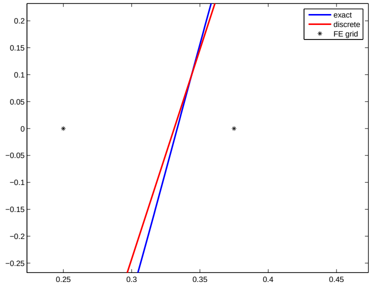
$$\|u - u_h\|_{L^1}, \|y - y_h\|, \|p - p_h\|_{L^\infty} \le C\left\{h^2 + \|p - R_h p\|_{L^\infty}\right\}$$

Sketch of proof:

- $\|u - u_h\|_{L^1} \le (b - a)\mathcal{L}(\{p > 0, p_h \le 0\} \cup \{p < 0, p_h \ge 0\})$
- $\{p > 0, p_h \le 0\} \cup \{p < 0, p_h \ge 0\} \subseteq \{|p(x)| \le \|p - p_h\|_\infty\} \Rightarrow$
- $\|u - u_h\|_{L^1} \le C\|p - p_h\|_\infty$
- $\|p - p_h\|_\infty \le \|p - R_h p\|_\infty + \|R_h p - p_h\|_\infty$
- $\|R_h p - p_h\|_\infty \le C\|y - y_h\|$.
- Combine these estimates with $(p, u_h - u) \ge 0$ and $(p_h, u - u_h) \ge 0$ (note that u is admissible as testfunction for the discrete problem!).

Numerical example with 2 switching points





Experimental order of convergence:

- **Active set 3.00073491, (here ≈) $\|u - u_h\|_{L^1}$: 3.00077834**
- **Function values 1.99966106**
- **$\|p - p_h\|_{L^\infty}$: 1.99979367**
- **$\|y - y_h\|_{L^\infty}$: 1.9997965**
- **$\|p - p_h\|_{L^2}$: 1.99945711**

Thank you very much for your attention!

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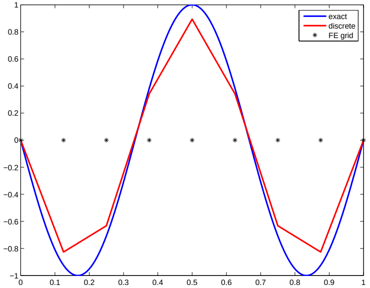

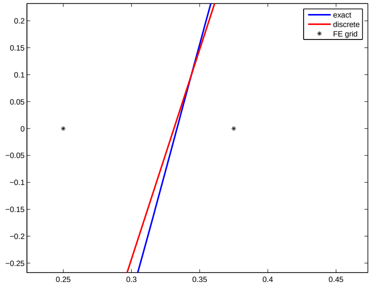

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