Risk Averse Shape Optimization

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Workshop on PDE Constrained Optimization of Certain and Uncertain Processes
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Outline

Conceptual sketch

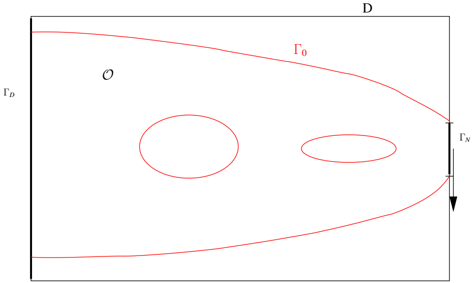


Figure: General setting in 2D

Optimization Task

$$\min_{\mathcal{O} \in \mathcal{O}_{ad}} J(\mathcal{O}) \qquad \mathcal{O}_{ad} = \{\mathcal{O} \subset D : \ \partial\mathcal{O} \text{ Lipschitz-continuous } \}$$

Linear elasticity model

The displacement u is given by the equation system

PDE

$$\begin{cases} -\text{div}(Ae(u)) = f & \text{in } \mathcal{O}, \\ u = 0 & \text{on } \Gamma_D, \\ (Ae(u))n = g & \text{on } \Gamma_N \end{cases}$$

- Elastic body $\mathcal{O} \subset \mathbb{R}^3$

$$\partial\mathcal{O} = \Gamma_N \cup \Gamma_D, \ \Gamma_D \neq \emptyset$$

- Volume forces f in $\mathcal{O}$
- Neumann forces g on Γ_N

where $e(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ is the linearized strain tensor

and Hooke's law

$$A\xi = 2\mu\xi + \lambda(\text{tr}\xi)\text{Id}, \ \text{for any symmetric matrix } \xi$$

Shape optimization problem

- Compliance

$$J(\mathcal{O}) = \int_{\mathcal{O}} f \cdot u \, dx + \int_{\Gamma_N} g \cdot u \, ds$$

- Least square error compared to target displacement

$$J(\mathcal{O}) = \left(\int_{\mathcal{O}} |u - u_0|^2 \, dx \right)^{\frac{1}{2}}$$

Shape optimization problem

$$\min_{\mathcal{O} \in \mathcal{O}_{ad}} \quad J(\mathcal{O}) \; + \; lV(\mathcal{O}) \qquad \text{with } l \; \in \mathbb{R}, \; l > 0$$

$$\mathcal{O}_{ad} = \{ \mathcal{O} \subset D : \; \partial\mathcal{O} \text{ Lipschitz-continuous } \}$$

Outline

Level set formulation

Implicit description of the domain $\mathcal{O}$ via a level set function ϕ

$$\begin{cases} \phi(x) = 0 <=> x \in \partial\mathcal{O} \\ \phi(x) < 0 <=> x \in \mathcal{O} \\ \phi(x) > 0 <=> x \notin \mathcal{O} \end{cases}$$

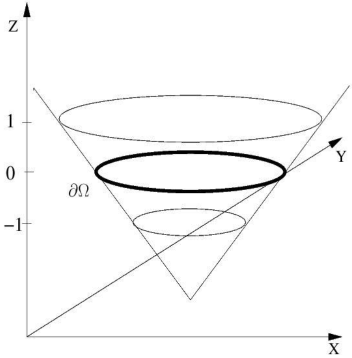


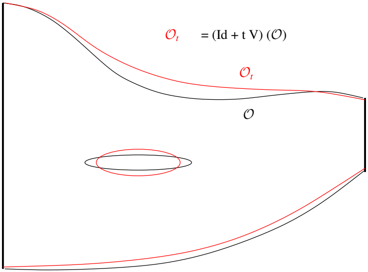
Figure: Levelset description in 2D

Outline

Shape gradient

We consider variations $\mathcal{O}_t = (Id + t \cdot V)(\mathcal{O})\,, \quad t > 0$ of a smooth elastic domain $\mathcal{O}$ for a smooth vector field V defined on the working domain D.
The shape derivative of $J(\mathcal{O})$ at $\mathcal{O}$ in direction V is defined as the Fréchet derivative of the mapping t $\rightarrow J(\mathcal{O}_t)$, i.e.

$$J(\mathcal{O}_t) = J(\mathcal{O}) + \left\langle \frac{\partial J}{\partial \mathcal{O}}, V \right\rangle + o(\|V\|)$$



cf. [Sokolowski, Zolesio '92], [Delfour, Zolesio '01]

Shape gradient

As a classical result of the shape sensitivity analysis the shape derivative takes the form

$$< \frac{\partial J}{\partial \mathcal{O}}, V > \; = \int_{\Gamma_N} \left(2 \left[\frac{\partial (g \cdot u)}{\partial n} \; + \; hg \cdot u \; + \; f \cdot u \right] \; - \; \mathcal{A}\epsilon(u) : \epsilon(u) \right) \; V \cdot \vec{n} \, d\nu$$

$$+ \int_{\Gamma_D} \left(\mathcal{A}\epsilon(u) : \epsilon(u) \right) \; V \cdot \vec{n} \, d\nu$$

Here h denotes the mean curvature of $\partial \mathcal{O}$ and $\vec{n}$ the outer normal.

Shape gradient in level set formulation

When the domain $\mathcal{O}$ is implicitly deformed by varying the level set function ϕ

$$\phi_t \;=\; \phi \;+\; t\psi$$

the level set equation

$$\partial_t\phi + |\nabla\phi|\, v\cdot n \;=\; 0 \qquad n = \frac{\nabla\phi}{|\nabla\phi|}$$

allows to define

$$< \frac{\partial J}{\partial\phi}, \psi >:=< \frac{\partial J}{\partial\mathcal{O}}, -\psi\cdot\frac{\vec{n}}{\|\nabla\,\phi\|} >$$

cf. [Osher, Sethian '88],

[Burger, Osher '04]

Shape gradient in level set formulation

We take into account a regularized gradient descent, based on the metric

$$\mathcal{G}(\theta,\zeta) = \int_D \theta\zeta + \frac{\sigma^2}{2}\nabla\theta\cdot\nabla\zeta\,\mathrm{d}x$$

which is related to a Gaussian filter with width σ.
The shape gradient is the solution of equation

$$\mathcal{G}(grad_\phi J,\theta) = <\frac{\partial J}{\partial\mathcal{O}},\theta> \qquad \forall\theta\in H_0^{1,2}(\mathcal{D})$$

Outline

Topological Derivative

Asymptotic behavier for infinitesimal small hole

$$\mathcal{T}(x) = \lim_{\rho \downarrow 0} \frac{J(\mathcal{O} \setminus \overline{B_\rho(x)}) - J(\mathcal{O})}{|\overline{B_\rho(x)}|}$$

Toppological derivative for the compliance

$$\mathbf{D}_{\text{topo}}\mathbf{J}(x) = \frac{\pi\,(\lambda + 2\mu)}{2\mu\,(\lambda + \mu)} \{4\mu A e(u_i) : e(u_i) + (\lambda - \mu) tr\, A e(u_i) tr\, e(u_i)\}$$

Outline

Optimization algorithm

time continuous regularized gradient descent:

$$\partial\phi(t) = -grad_\phi J(\phi)$$

with time discrete relaxation :

$$\mathcal{G}(\phi^{k+1} - \phi^k, \theta) = -\tau \ < \frac{\partial J}{\partial \mathcal{O}}, \theta > \qquad \forall \theta \in H_0^{1,2}(\mathcal{D})$$

additional ingrediens of the algorithm :

- multigrid method for the primal and the dual problem $(d = 3)$
- preconditioned CG $(d = 2)$
- cascadic optimization (from coarse to fine grid resolution)
- morphological smoothing when switching the grid resolution $(\sigma = 2.5h \ \text{ or } \ 4.5h)$
- topological changes are performed every 10 steps

Outline

Structure of Random Forces

Assume that ω follows a discrete distribution with scenarios ω_σ and probabilities π_σ with $\sum_{\sigma=1}^{S} \pi_\sigma = 1$ and 'basis' loads (f^k, g^m) spanning the load space:

$$f(\omega) = \sum_{k=1}^{K} \alpha_k f^k , \qquad g(\omega) = \sum_{m=1}^{M} \beta_m f^m$$

by linearity :

$$\bar{u}(\mathcal{O}, \omega) = \sum_{k=1}^{K} \alpha_k \, u_f^k + \sum_{m=1}^{M} \beta_m u_g^m$$

$$\text{solves} \qquad A(\mathcal{O}, \bar{u}(\mathcal{O}, \omega_\sigma), \varphi) = l(\mathcal{O}, \varphi, \omega_\sigma)$$

Outline

Expected Excess

Considering the following problem

$$\min\left\{\sum_{\sigma=1}^{S}\pi_\sigma max\{J(\mathcal{O},\omega_\sigma)-\eta,0\}\quad \mathcal{O}\in\mathcal{U}_{ad}\right\}$$

and the smooth approximation of the maximun function

$$\max\{a,0\}=\frac{\sqrt{a^2}+a}{2}\approx\frac{\sqrt{a^2+\epsilon}+a}{2}=:Max(a)\qquad \epsilon>0$$

we get the differentiable Expected Excess functional.

expected excess

$$\min\{\mathbb{EE}(\mathcal{O}):=\sum_{\sigma=1}^{S}\pi_\sigma\, Max(J(\mathcal{O},\omega_\sigma))\quad :\mathcal{O}\in\mathcal{U}\}$$

Outline

Excess Probability

Now consider

$$\min\{\mathbb{P}(J(\mathcal{O},\omega) > \eta) \quad : \mathcal{O} \in \mathcal{U}_{ad}\}$$

The random stage ω follows a discrete distribution with finitely many scenarios ω_σ and probabilities π_σ according to $\sum_{\sigma=1}^{S} \pi_\sigma = 1$ and we get

$$\min\{\mathbb{P}(J(\mathcal{O},\omega_\sigma) > \eta) = \sum_{\sigma=1}^{S} \pi_\sigma \, H(J(\mathcal{O},\omega_\sigma) - \eta)$$

where H(x) is supposed to be the Heaviside function. Again we use a smooth approximation H(x) $\approx \frac{1}{2} + \frac{1}{2} tanh(kx) = \frac{1}{1+e^{-2k}}$

excess probability

$$\min\{\mathbb{EP}(\mathcal{O}) := \sum_{\sigma=1}^{S} \pi_\sigma \, H(J(\mathcal{O},\omega_\sigma)) \quad : \mathcal{O} \in \mathcal{U}\}$$

Outline

Test Setting

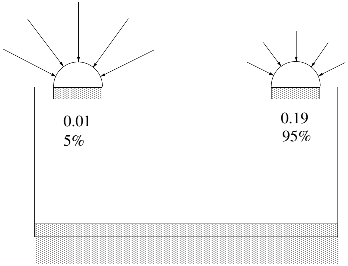


Figure: Nonsymmetric stochastic loading

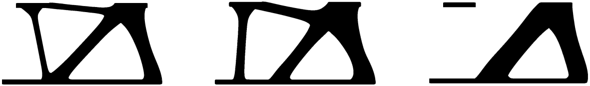


Figure: The results for EV, EE and EP; the threshold η is set to 0.4

Expected Exccess

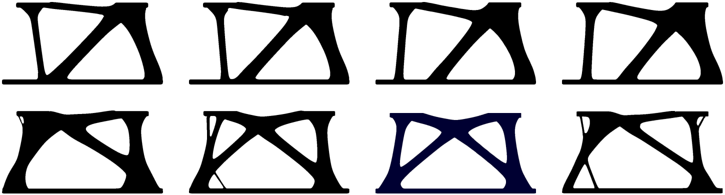
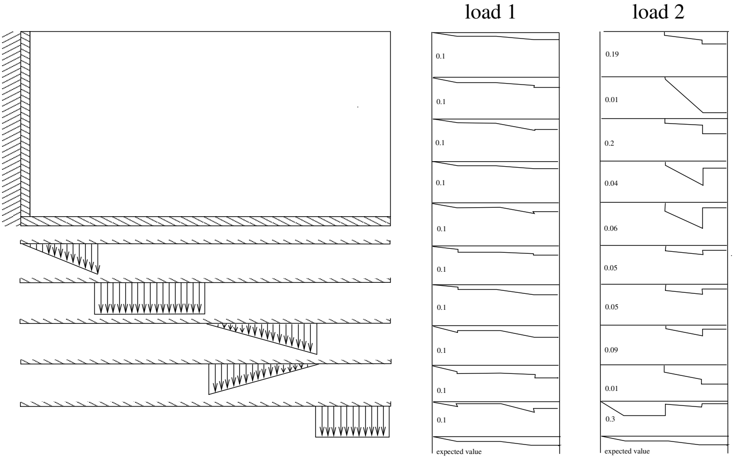


Figure: A sequence of results for the optimization with respect to the expected excess for $\eta = 0.1, 0.2, 0.3, 0.4, 0.6, 0.8, 1.0. , 1.5$.

Cantilever


load 1
load 2
0.1
0.1
0.1
0.1
0.1
0.1
0.1
0.1
0.1
0.1
expected value
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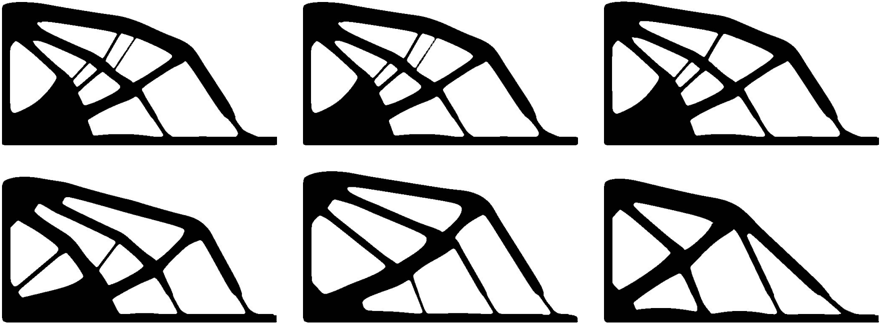
0.01
0.2
0.04
0.06
0.05
0.05
0.09
0.01
0.3
expected value



Thank you !