# Calibration with Reduced Order Models for PIDEs

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GAMM Workshop
on PDE Constrained Optimization
2009

June 5th, 2009

# What's this presentation about?

goal: calibration of option pricing models

- model prices are calculated via a partial (integro) differential equation

problem: no closed-form solution for many of these problems

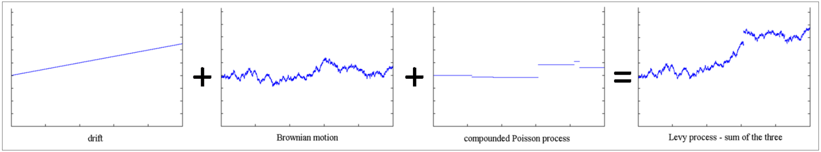
- a numerical solution is required
- multiple solution of large-scaled linear systems of equations (LSE)
- calibration problems are very time expensive

solution: Proper Orthogonal Decomposition (POD)

- save computing time during the repeated solution of the LSE's

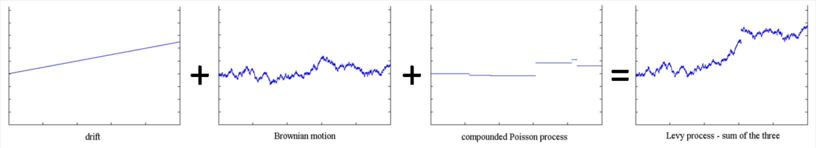
# Pricing european options in jump diffusion models

Typical path of a jump diffusion process



# Pricing european options in jump diffusion models

Typical path of a jump diffusion process




## Log-transformed PIDE of a general jump diffusion model

$$D_T(x,T) - \frac{\sigma^2(x,T)}{2} D_{xx}(x,T) + \left( r(T) + \frac{\sigma^2(x,T)}{2} - \lambda\zeta \right) D_x(x,T) +$$

$$+ \lambda(1+\zeta) D(x,T) - \lambda \int_{-\infty}^{+\infty} D(x-y,T) e^y \, f(y) dy = 0$$

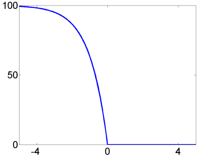
$$\text{on } (-\infty,\infty) \times (0, T_{max})$$

with initial condition $D(x,0) = \max\{S_0 - S_0 e^x, 0\}$.

# Problem: Integrability of the initial condition

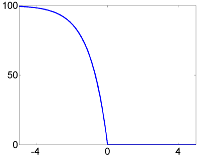
$D(x,0) = \max\{S_0 - S_0 e^x, 0\} =: D_0(x)$

$D_0(x) \overset{x \to -\infty}{\longrightarrow} S_0 \Rightarrow$ not $L^2(\boldsymbol{R})$-integrable



# Problem: Integrability of the initial condition

$$D(x,0) = \max\{S_0 - S_0 e^x, 0\} =: D_0(x)$$
$$D_0(x) \overset{x\to-\infty}{\longrightarrow} S_0 \Rightarrow \text{not } L^2(\boldsymbol{R})\text{-integrable}$$




## Weighted function spaces ($\mu \in \boldsymbol{R}$)

$$L^2_{-\mu}(\boldsymbol{R}) := \left\{ v \in L^1_{loc}(\boldsymbol{R}) : v(\cdot)e^{-\mu|\cdot|} \in L^2(\boldsymbol{R}) \right\}$$

with $\langle v, w \rangle_{L^2_{-\mu}} := \int_R v(x)w(x)e^{-2\mu|x|}dx$ a Hilbert space.

$$H^1_{-\mu}(\boldsymbol{R}) := \left\{ v \in L^1_{loc}(\boldsymbol{R}) : v(\cdot)e^{-\mu|\cdot|}, v'(\cdot)e^{-\mu|\cdot|} \in L^2(\boldsymbol{R}) \right\}$$

with $\langle v, w \rangle_{H^1_{-\mu}} := \langle v, w \rangle_{L^2_{-\mu}} + \langle v', w' \rangle_{L^2_{-\mu}}$ a Hilbert space, too.

# Variational formulation

With an appropriate bilinear form

$$a^{-\mu}(\cdot;\ \cdot,\cdot) : [0, T_{max}] \times \left(H^1_{-\mu}(\boldsymbol{R}) \times H^1_{-\mu}(\boldsymbol{R})\right) \to \boldsymbol{R}$$

we get the variational formulation:

## Problem

Find $D \in W([0, T_{max}], H^1_{-\mu}(\boldsymbol{R}))$ with

$$\left\langle D_T(\cdot, T), w(\cdot)\right\rangle_{L^2_{-\mu}} + a^{-\mu}\left(T; D(\cdot, T), w(\cdot)\right) = 0 \quad \forall\ w \in H^1_{-\mu}(\boldsymbol{R})$$

with initial condition:

$$\left\langle D(\cdot, 0), w(\cdot)\right\rangle_{L^2_{-\mu}} = \left\langle D_0(\cdot), w(\cdot)\right\rangle_{L^2_{-\mu}} \ \forall\ w \in H^1_{-\mu}(\boldsymbol{R})$$

# Time dependent bilinear form $a$

$$a^{-\mu}(T;\ v, w) = \int\limits_R \frac{\sigma^2(x, T)}{2} v'(x) w'(x) e^{-2\mu|x|} dx$$

$$+ \int\limits_R \left( r(T) + \frac{\sigma^2(x, T)}{2} - \lambda\zeta + \left(\frac{\sigma(x, T)}{2}\right)_x + \sigma^2(x, T) \cdot \mu \cdot sign(x) \right) \cdot v'(x) w(x) e^{-2\mu|x|} dx$$

$$+ \int\limits_R \lambda(1+\zeta) v(x) w(x) e^{-2\mu|x|} dx - \lambda \int\limits_R \int\limits_R v(x-y) w(x) e^{-2\mu|x|} e^y f(y) dy\, dx$$

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$$+ \int\limits_R \lambda(1+\zeta)v(x)w(x)e^{-2\mu|x|}dx - \ \lambda \int\limits_R \int\limits_R v(x-y)w(x)e^{-2\mu|x|}e^y f(y)dy\ dx$$

**Properties of $a$ ($\exists\ c_1, c_2 > 0,\ c_3 \in \boldsymbol{R}$)**

$$|a^{-\mu}(T;\ v,w)| \leq c_1 \cdot ||v||_{H^1_{-\mu}} ||w||_{H^1_{-\mu}} \qquad \forall\ T \in [0,\ T_{max}]$$

$$a^{-\mu}(T;\ v,v) \geq c_2 \cdot ||v||^2_{H^1_{-\mu}} - c_3 \cdot ||v||^2_{L^2_{-\mu}} \qquad \forall\ T \in [0,\ T_{max}]$$

# Spatial discretization

$\mathcal{H}_n \overset{n\to\infty}{\longrightarrow} H^1_{-\mu}$ with a finite basis $\{\Phi_i\}_{i=1}^n$
$\Rightarrow \hat{D}(x,T) = \sum_{i=1}^n \alpha_i(T)\Phi_i(x)$

## Semi-discretized problem

Find $\alpha_i(\cdot) : (0, T_{max}) \to \boldsymbol{R}$ , $i = 1, \ldots, n$

$$\sum_{i=1}^n \dot{\alpha}_i(T)\left\langle \Phi_i, \Phi_j \right\rangle_{L^2_{-\mu}} + \sum_{i=1}^n \alpha_i(T) a^{-\mu}\left(T; \Phi_i, \Phi_j\right) = 0$$

with initial condition: $\forall\, j = 1, \ldots, n$

$$\sum_{i=1}^n \alpha_i(0)\left\langle \Phi_i, \Phi_j \right\rangle_{L^2_{-\mu}} = \left\langle D_0(\cdot), \Phi_j \right\rangle_{L^2_{-\mu}} \;\forall\, j = 1, \ldots, n$$

# Numerical solution

## Linear system of equations

Find $\alpha(T_k) \in \mathbb{R}^n \ \forall\ k = 0, \ldots, m$

$$\Big(M + \Delta T \cdot \theta \cdot A(T_{k+1})\Big) \cdot \alpha(T_{k+1}) =$$
$$= \Big(M - \Delta T \cdot (1-\theta) \cdot A(T_k)\Big) \cdot \alpha(T_k)$$

with initial condition $M \cdot \alpha(T_0) = B$

- $\theta = 0$: explicit Euler method
- $\theta = 0.5$: Crank-Nicolson method
- $\theta = 1$: implicit Euler method

- $M$ sparse mass matrix
- $A(T)$ stiffness matrix (dense due to the integral term)

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# Proper Orthogonal Decomposition

What is POD?

Given: Functions/vectors $u_1, \ldots, u_n$ ($\Rightarrow$ Snapshots)
$\mathcal{V} := span\{u_1, \ldots, u_n\}$ with $rg(\mathcal{V}) = r \leq n$

Find: Orthonormal basis function/vectors $\Psi_1, \ldots, \Psi_p$ (with $p \leq r$)
with which an „average“ function $v \in \mathcal{V}$ can be described at the best

$\Rightarrow$ Extract significant information from given data

# Illustration


Holiday picture

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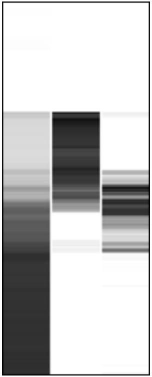
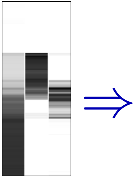
Holiday picture

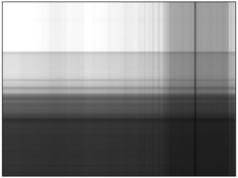
$\Longrightarrow$



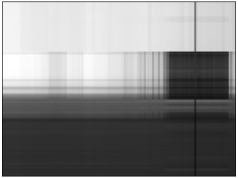
Extract significant information

# Illustration

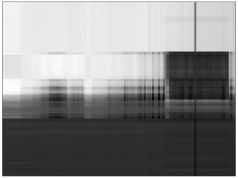




1 basis function



2 basis function



3 basis function

10 basis functions

20 basis functions

50 basis functions

# Definition

Let „snapshots" $u_1, \dots, u_n \in H$ be given.

## POD basis

Find orthonormal vectors $\Psi_1, \dots, \Psi_r \in span(u_1, \dots, u_n)$, solving:

$$\min_{\Psi_1,\dots,\Psi_p} \sum_{i=1}^{n} \gamma_i \left|\left| u_i - \sum_{j=1}^{p} \langle u_i, \Psi_j \rangle_H \Psi_j \right|\right|_H^2$$
$$s.t. \quad \langle \Psi_k, \Psi_l \rangle_H = \delta_{kl} \quad \forall k, l = 1, \dots, p$$

for all $p \in \{1, \dots, r\}$ and weighting factors $\gamma_i > 0$, $i = 1, \dots, n$.

e.g. $\gamma_i = \frac{1}{n}$ for the arithmetic mean.

# Calculation of the basis functions

## Sufficient optimality condition

Define $\mathcal{R} \in \mathcal{L}(H,H)$ with $\mathcal{R}z := \sum_{i=1}^{n} \gamma_i \langle z, u_i \rangle_H u_i$
Then there exists a complete orthonormal basis $\{\Psi_k\}_{k\geq 1}$ and a sequence of non-negative real numbers $\{\lambda_k\}_{k\geq 1}$ with

$$\mathcal{R}\Psi_k = \lambda_k \Psi_k \quad \text{mit } \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0 \text{ und } \lambda_j = 0 \; \forall j > r$$

The eigenvectors $\Psi_k$ solve the minimization problem above.
The eigenvectors corresponding to the $p$ largest eigenvalues, build the POD basis of rank $p$.

# Using POD to solve the PIDE

1. Snapshots: $D(x, T_k) \in \mathcal{H}_n$ $(k = 0, \ldots, m)$ from a known solution

2. Calculate the POD basis $\{\Psi_i\}_{i=1}^{p}$ with rank $p$
   (by solving the eigenvalue problem)
   $\Rightarrow \exists\ \beta(T_k)$ with $D(x, T_k) \approx \sum_{i=1}^{p} \beta(T_k)\Psi_i(x)$

3. Use $\{\Psi_i\}_{i=1}^{p}$ instead of the FE basis functions

# Reminder: Discretization in $x$ direction

$\mathcal{H}_n$ with basis $\{\Phi_i\}_{i=1}^n$
$\Rightarrow D(x,T) = \sum_{i=1}^n \alpha_i(T)\Phi_i(x)$

## Semi discretized problem

Find $\alpha_i(\cdot) : (0, T_{max}) \to \boldsymbol{R}$ , $i = 1, \ldots, n$, where

$$\sum_{i=1}^{n} \dot{\alpha}_i(T)\Big\langle \Phi_i, \Phi_j \Big\rangle_{L^2_{-\mu}} + \sum_{i=1}^{n} \alpha_i(T) a^{-\mu}\Big(T; \Phi_i, \Phi_j\Big) = 0$$

with initial condition: $\forall\, j = 1, \ldots, n$

$$\sum_{i=1}^{n} \alpha_i(0)\Big\langle \Phi_i, \Phi_j \Big\rangle_{L^2_{-\mu}} = \Big\langle D_0(\cdot), \Phi_j \Big\rangle_{L^2_{-\mu}} \;\forall\, j = 1, \ldots, n$$

# Discretization in $x$ direction using the POD basis

$\mathcal{V}^p$ with basis $\{\Psi_i\}_{i=1}^p$
$\Rightarrow D^{POD}(x,T) = \sum_{i=1}^p \alpha_{POD,i}(T)\Psi_i(x)$

## Semi discretized problem

Find $\alpha_{POD,i}(\cdot) : (0, T_{max}) \to \boldsymbol{R}$ , $i = 1, \dots, p$, where

$$\sum_{i=1}^p \dot{\alpha}_{POD,i}(T)\Big\langle \Psi_i, \Psi_j \Big\rangle_{L^2_{-\mu}} + \sum_{i=1}^p \alpha_{POD,i}(T) a^{-\mu}\Big(T; \Psi_i, \Psi_j\Big) = 0$$

with initial condition: $\forall\, j = 1, \dots, p$

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# Comparison POD vs. FEM

For each time step $T_k$ - i.e. $m$ times - a linear system of equations has to be solved:

FEM LSE on the scale of $n \times n$
($n$: number of discretization steps in $x$ direction)

POD LSE on the scale of $p \times p$
($p$: number of POD basis functions)
where $p \ll n$

# Comparison POD vs. FEM

Use FEM solution $u_1^{FEM}(x), \ldots, u_n^{FEM}(x)$ as snapshots to build a POD basis $u_1^{POD}(x), \ldots, u_n^{POD}(x)$ the solution of the corresponding POD system.

## Error estimation

1. Implicit Euler

$$\frac{1}{n}\sum_{i=1}^{n}\left|\left|u_i^{FEM} - u_i^{POD}\right|\right|_H^2 \leq \bar{C}_1 \sum_{j=p+1}^{r} \lambda_j$$

2. Crank-Nicolson method, with $\Delta t < \sqrt[3]{\frac{16\kappa}{c_{lip}^2 \alpha ||S||_2^2}}$

$$\frac{1}{n}\sum_{i=1}^{n}\left|\left|u_i^{FEM} - u_i^{POD}\right|\right|_H^2 \leq \tilde{C}_1 \sum_{j=p+1}^{r} \lambda_j$$

# Comparison POD vs. FEM: numerical results

- $\epsilon = \frac{1}{n} \sum_{i=1}^{n} \left|\left| u_i^{FEM} - u_i^{POD} \right|\right|_H^2$, $\lambda_i$ eigenvalues
- $p$ number of POD basis functions

| $p$ | $\epsilon$ | $\sum_{i=p+1}^{r} \lambda_i$ |
|---|---|---|
| 5 | 5.95e-002 | 2.42e+001 |
| 6 | 7.71e-003 | 3.99e+000 |
| 7 | 9.38e-004 | 6.35e-001 |
| 8 | 1.12e-004 | 9.70e-002 |
| 9 | 1.32e-005 | 1.43e-002 |
| 10 | 1.53e-006 | 2.02e-003 |
| 11 | 1.74e-007 | 2.74e-004 |
| 12 | 1.95e-008 | 3.59e-005 |

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# Calibration of option pricing models

**Calibration problem (Merton's model)**

Find $(\sigma, \lambda, \mu_J, \sigma_J)$ as a solution of

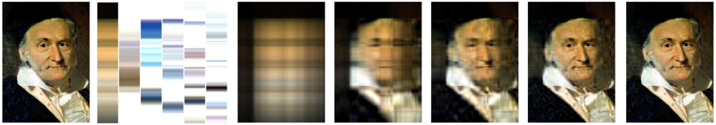
$$\min_{(\sigma,\lambda,\mu_J,\sigma_J)} \sum_{i=1}^{k} \Big( D((\sigma, \lambda, \mu_J, \sigma_J);\; x_i, T_i) - C_M^i \Big)^2$$

$C_M^i$ given market prices for different options with maturities $T_i$ and strike prices $x_i$

$D((\sigma, \lambda, \mu_J, \sigma_J);\; x_i, T_i)$ corresponding model prices determined by the Dupire PIDE

# Necessity of updates

Rebuild Gauss using information of Gauss



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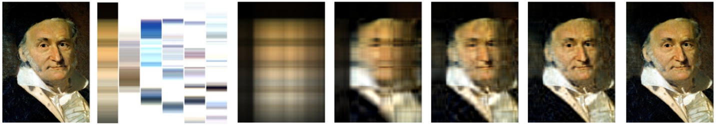

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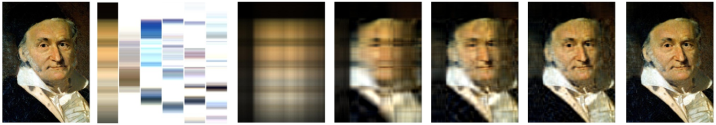

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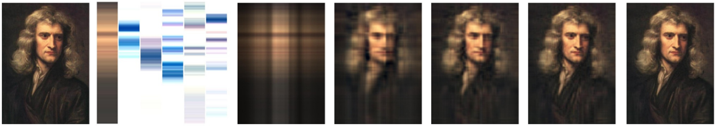
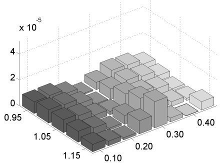


# Trust region updates - Example (Merton's Jump diffusion model)

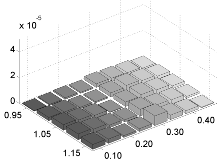
Line search algorithm using the FEM solutions:

FEM evaluations: 285 (2000 $\times$ 2000)
Time overall: 955 sec



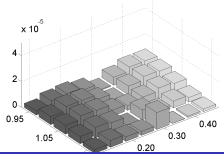

Trust region POD:

FEM evaluations: 9 (2000 $\times$ 2000)
POD evaluations: 328
Time overall: 60 sec (FEM: 36 sec, POD: 24 sec)




Multi-level trust region POD:

FEM evaluations: 3 (500 $\times$ 500), 2 (1000 $\times$ 1000) , 5 (2000 $\times$ 2000)
POD evaluations: 321
Time overall: 42 sec (FEM: 22 sec, POD: 20 sec)



Thank you
for your attention!