Drag minimization for compressible Navier-Stokes equations

Jan Sokolowski, IECN Nancy
joint work with Pavel I. Plotnikov, Novosibirsk,
and Antoni Zochowski, Warsaw

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Plan

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- Flow around an airfoil in wind tunnel - drag functional
- Compressible N-S equations: well posedness and sensitivity analysis
- Material derivatives of approximate solutions
- Shape derivatives of drag functional
- Numerical results in two spatial dimensions

Results

- Inhomogenuous boundary value problems in bounded domains for Navier-Stokes-Fourier equations (JMPA, 2009)
- Existence of solutions and asymptotic analysis (Elsevier, 2008)
- Stationary solutions of Navier-Stokes equations for diatomic gases (UMN, 2007)
- Approximate solutions for N-S boundary value problems; Shape sensitivity analysis and the differentiability of the drag functional (SIMA, 2008)
- Shape gradient of drag functional (IECN, 2009)
- Numerical results (AIMS, 2008, MMAR, 2009, CFD-MIT, 2009)

Example

Let us consider the bounded domain Ω with the smooth boundary. Take the state equation in the form

$$-\Delta u(x) = F(x, u(x)) \quad \text{in} \quad \Omega, \qquad u(x) = u_0 \equiv \text{const.} \quad \text{on} \quad \partial\Omega\,.$$

The shape optimization problem is an optimal choice of the domain within an admissible class, in such a way that the boundary shape functional defined on $\partial\Omega$ is minimized. The functional is selected in the form which looks like the *drag*, i.e.,

$$j(\partial\Omega) = u_0 \int_{\partial\Omega} \nabla u(x) \cdot n(x) dS\,. \tag{1}$$

Using the Gauss formula we can rewrite this functional in the distributed form

$$j(\partial\Omega) = \int_{\Omega} (|\nabla u|^2 - uF(u, x))\, dx \tag{2}$$

Representation of such kind are common in viscous fluid dynamics since they have clear physical meaning. For instance, the gradient parts represents the rate of dissipation of the energy. However functional (2) is only weakly lower semicontinuous in the energy space and can be used mostly for minimization problems. The other approach employed in the paper can be describe in the following way. Introduce a smooth scalar function $\eta(x)$ such that $\eta(x) \equiv 1$ on $\partial\Omega$ and rewrite the expression for $j(\partial\Omega)$ in the equivalent form

$$J(\Omega) = u_0 \int_\Omega (\nabla\eta \cdot \nabla u - \eta F(x,u))\, dx \ . \tag{3}$$

This functional is weakly continuous, and its principle part is linear with respect to state variable u. The technical difficulty is that the integrand contains an arbitrary function η while the result is independent of η, and we have to eliminate the influence of η on the results of calculations.

Shape sensitivity analysis for drag minimization

- apply material derivatives technique for state equation and shape functional in the distributed form (3) with a function η
- use the structure theorem for the shape gradient obtained in the distributed form: show that the shape gradient is given in fact by a function $g(x), x \in \partial S$ on the boundary of the obstacle S
- pass to the singular limits in volume integrals and identify the boundary form of the shape gradient, i.e., the function g, show that the geometrical part of the shape derivative of the drag functional in fact vanishes
- test the expression for the shape gradient by numerical drag minimization

Flow around an airfoil in wind tunnel

For N-S equations the stress tensor is equal to

$$\mathbb{T} =: \nabla \mathbf{u} + \nabla \mathbf{u}^* + (\lambda - 1)\,\mathrm{div}\,\mathbf{u}\,\mathbf{I} - \frac{R}{\epsilon^2} p\mathbf{I}\,,$$

and the hydrodynamical force acting on the body (obstacle) S is equal

$$\mathbf{J}(S) =: -\int_{\partial S} \mathbb{T}\mathbf{n}\,ds$$

$$= -\int_{\partial S} (\nabla \mathbf{u} + (\nabla \mathbf{u})^* + (\lambda - 1)\,\mathrm{div}\,\mathbf{u}\mathbf{I} - \frac{R}{\epsilon^2} p\mathbf{I}) \cdot \mathbf{n} ds\,.$$

Drag for flow around an airfoil

$$\int_{\partial S} \mathbb{T}\mathbf{n}\, ds = \int_{\Omega} (\eta \,\mathrm{div}\, \mathbb{T} + \mathbb{T}\nabla\eta)\, dx, \qquad \mathrm{div}\, \mathbb{T} = R\varrho\mathbf{u}\nabla\mathbf{u}$$

using the above identities we obtain (η is given as before !)

$$\mathbf{J}(S) =: -R \int_{\Omega} \eta\varrho\mathbf{u}\nabla\mathbf{u}\, dx$$

$$-\int_{\Omega} (\nabla\mathbf{u} + (\nabla\mathbf{u})^* + (\lambda - 1)\,\mathrm{div}\,\mathbf{u}\mathbf{I} - \frac{R}{\delta} p\mathbf{I})\nabla\eta\, dx$$

The drag J_D is a work in unit time developed by the component of **J** parallel to the airfoil speed **U**,

$$J_D(S) = \mathbf{U} \cdot \mathbf{J}(S).$$

Shape optimization

In general the mathematical analysis of shape optimization problems includes the following steps, with the mathematical proofs of the required facts,

- existence of solutions,
- uniqueness and optimality conditions,
- numerical method of solution.

Obstacle in bounded domain

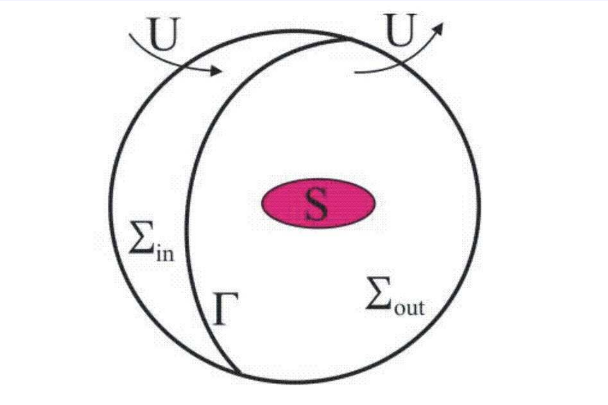

Figure: Flow domain $\Omega = B \setminus \overline{S}$.

Drag functional

One of the main applications of the theory of compressible viscous flows is the optimal shape design in aerodynamics, e.g., the minimization of the drag of airfoil travelling in atmosphere with uniform speed $\mathbf{U}_\infty$. The hydro-dynamical force acting on the body S is defined by

$$\mathbf{J}(S) = -\int_{\partial S} (\nabla \mathbf{u} + (\nabla \mathbf{u})^* + (\lambda - 1)\,\mathrm{div}\,\mathbf{u}\mathbf{I} - \frac{R}{\delta} p\mathbf{I}) \cdot \mathbf{n} dS \,.$$

In a frame attached to the moving body the drag is the component of $\mathbf{J}$ parallel to $\mathbf{U}_\infty$,

$$J_D(S) = \mathbf{U}_\infty \cdot \mathbf{J}(S).$$

For the fixed data, the drag can be regarded as a functional depending on the shape of the obstacle S.

Geometry of flow domain

We assume that the viscous gas occupies the double-connected domain $\Omega = B\setminus\overline{S}$, where $B \subset \mathbb{R}^3$, is a hold-all domain with the smooth boundary $\Sigma = \partial B$, and $S \subset B$ is a compact obstacle. Furthermore, we assume that the velocity of the gas coincides with a given vector field $\mathbf{U} \in C^\infty(\mathbb{R}^3)^3$ on the surface Σ. In this framework, the boundary of the flow domain Ω is divided into the three subsets, inlet $\Sigma_{\rm in}$, outgoing set $\Sigma_{\rm out}$, and characteristic set Σ_0, which are defined by the equalities $\Sigma_{\rm in} = \{x \in \Sigma : \mathbf{U}\cdot\mathbf{n} < 0\}$, $\Sigma_{\rm out} = \{x \in \Sigma : \mathbf{U}\cdot\mathbf{n} > 0\}$, $\Sigma_0 = \{x \in \partial\Omega : \mathbf{U}\cdot\mathbf{n} = 0\}$, where $\mathbf{n}$ stands for the outward normal to $\partial\Omega = \Sigma \cup \partial S$. In its turn the compact $\Gamma = \Sigma_0 \cap \Sigma$ splits the surface Σ into three disjoint parts $\Sigma = \Sigma_{\rm in} \cup \Sigma_{\rm out} \cup \Gamma$.

Generalized solutions of compressible N-S equations

For a given geometry of the flow domain the problem is to find the velocity field **u** and the gas density ϱ satisfying the following equations along with the boundary conditions

$$\Delta \mathbf{u} + \lambda \nabla \operatorname{div} \mathbf{u} = R\varrho \mathbf{u} \cdot \nabla \mathbf{u} + \frac{R}{\delta} \nabla p(\varrho) \text{ in } \Omega,$$

$$\operatorname{div}(\varrho \mathbf{u}) = 0 \text{ in } \Omega,$$

$$\mathbf{u} = \mathbf{U} \text{ on } \Sigma, \quad \mathbf{u} = 0 \text{ on } \partial S,$$

$$\varrho = \varrho_0 \text{ on } \Sigma_{\mathrm{in}},$$

where the pressure $p = p(\varrho)$ is a smooth, strictly monotone function of the density, $\epsilon = \sqrt{\delta}$ is the Mach number, R is the Reynolds number, λ is the viscosity ratio, and ϱ_0 is a positive constant.

Shape sensitivity analysis of approximate solutions

The analysis is performed in the following way (SIMA, 2008).

- Solutions of *compressible perturbations* of the Stokes boundary value problem are called approximate solutions of our problem. The boundary value problem for the approximate solutions is defined in an appropriate way.
- The shape perturbations $\Omega_\varepsilon = B \setminus S_\varepsilon$ are introduced, and the boundary value problem in variable domain is transported to the fixed domain Ω. The existence of weak material derivatives for the approximate solutions is shown.
- The adjoint state is defined and the shape differentiability of the drag functional is shown. It is shown that the shape gradient is given by a function. Therefore, the levelset method can be used for numerical solution.

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Geometrical domain perturbations

We start with description of our framework for shape sensitivity analysis, or more general, for well-posedness of compressible N-S equations. To this end we choose the vector field $\mathbf{T} \in C^2(\mathbb{R}^3)^3$ vanishing in the vicinity of Σ, and define the mapping

$$y = x + \varepsilon \mathbf{T}(x),$$

which describes the perturbation of the shape of the obstacle. Once, the result on differentiability of solutions to N-S equations with respect to $\varepsilon \to 0$ is proved, we consider the trace of the mapping $\mathbf{T}$ on the boundary ∂S of the obstacle S,

$$\mathbf{T}(x) = f(x)\mathbf{n}(x) \ \text{ for } \ x \in \partial S.$$

The function $f(x), x \in \partial S$ describes the boundary variations ∂S_ε in the normal direction $\mathbf{n}(x), x \in \partial S$.

Boundary shape gradient

We want to identify the shape gradient

$$g(x), \quad x \in \partial S$$

from the complicated expression obtained in terms of the material derivatives of the solutions to N-S equations:

$$\int_{\partial S} g(x) f(x) ds := \frac{d}{d\varepsilon} J_D(S_\varepsilon)\Big|_{\varepsilon=0} = L_e(\mathbf{T}) + L_u(\mathbf{w}, \omega, \psi)$$

where L_e, L_u are the geometrical and dynamical parts of the shape derivative $\frac{d}{d\varepsilon} J_D(S_\varepsilon)\Big|_{\varepsilon=0}$.

Boundary shape gradient

To this end the mapping $\mathbf{T}_\tau(x)$ is introduced in such a way that the mapping is an extension of the trace $f(x)\mathbf{n}(x)$ given on $x \in \partial S$, and with $\tau \to 0$ the support of $\mathbf{T}_\tau(x)$ tends to ∂S. In such a way $\int_{\partial S} g(x)f(x)ds$ can be obtained as the singular limit of the volume integral defined in Ω

$$L_e(\mathbf{T}_\tau) + L_u(\mathbf{w}_\tau, \omega_\tau, \psi_\tau) = \int_\Omega \mathfrak{F}_\tau(x)dx \to \int_{\partial S} g(x)f(x)ds$$

the functions $(\mathbf{w}_\tau, \omega_\tau, \psi_\tau) \to (\mathbf{h}, \varpi, \upsilon)$ are defined by the so-called adjoint system for the material derivatives, and in the limit for $\tau \to 0$ by the shape derivatives. In this way we can obtain also the boundary value problem for the shape derivatives for solutions of the direct problem as well as of the adjoint state.

Fixed domain setting

The solutions of NSE in the perturbed domain

$$\Omega_\varepsilon = \mathcal{T}_\varepsilon(\Omega) = (\mathbf{I} + \varepsilon \mathbf{T})(\Omega) =$$
$$\textit{normal shift of the obstacle boundary} = {}''\Omega + \varepsilon f \mathbf{n} \partial S''$$

are denoted by

$$\bar{\mathbf{u}}_\varepsilon(y) = \bar{\mathbf{u}}_\varepsilon(x + \varepsilon \mathbf{T}(x)), \quad \bar{\varrho}_\varepsilon(y) = \bar{\varrho}_\varepsilon(x + \varepsilon \mathbf{T}(x))$$

Fixed domain formulation

We introduce the functions $\mathbf{u}_\varepsilon(x)$ and $\varrho_\varepsilon(x)$ defined in the unperturbed domain Ω by the formulae

$$\mathbf{u}_\varepsilon(x) = \mathbf{N}\bar{\mathbf{u}}_\varepsilon(x + \varepsilon\mathbf{T}(x)), \quad \varrho_\varepsilon(x) = \bar{\varrho}_\varepsilon(x + \varepsilon\mathbf{T}(x)),$$

where

$$\mathbf{N}(x) = [\det\,(\mathbf{I} + \varepsilon\mathbf{T}'(x))(\mathbf{I} + \varepsilon\mathbf{T}'(x))]^{-1}.$$

is the adjugate matrix of the Jacobi matrix $\mathbf{I} + \varepsilon\mathbf{T}'$. Furthermore, we also use the notation $\mathfrak{g}(x) = \sqrt{\det \mathbf{N}}$. The matrices $\mathbf{N}(x)$ depends analytically upon the small parameter ε and

$$\mathbf{N} = \mathbf{I} + \varepsilon\mathbf{D}(x) + \varepsilon^2\mathbf{D}_1(\varepsilon, x),$$

where $\mathbf{D} = \mathrm{div}\,\mathbf{T}\mathbf{I} - \mathbf{T}'$.

Fixed domain formulation

For $\mathbf{u}_\varepsilon, \varrho_\varepsilon$, the following boundary value problem is obtained

$$\Delta \mathbf{u}_\varepsilon + \nabla\Big(\lambda \mathfrak{g}^{-1} \operatorname{div} \mathbf{u}_\varepsilon - \frac{R}{\delta} p(\varrho_\varepsilon)\Big) = \mathscr{A}\mathbf{u}_\varepsilon + R\mathscr{B}(\varrho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) \text{ in } \Omega,$$

$$\operatorname{div}\left(\varrho_\varepsilon \mathbf{u}_\varepsilon\right) = 0 \text{ in } \Omega,$$

$$\mathbf{u}_\varepsilon = \mathbf{U} \text{ on } \Sigma, \quad \mathbf{u}_\varepsilon = 0 \text{ on } \partial S,$$

$$\varrho_\varepsilon = \varrho_0 \text{ on } \Sigma_{\mathrm{in}}.$$

Notation

Here, the linear operator $\mathscr{A}$ and the nonlinear mapping $\mathscr{B}$ are defined in terms of **N**,

$$\mathscr{A}(\mathbf{u}) = \Delta\mathbf{u} - \mathbf{N}^{-1}\,\mathrm{div}\,\big(\mathfrak{g}^{-1}\mathbf{N}\mathbf{N}^{*}\nabla(\mathbf{N}^{-1}\mathbf{u})\big),$$

$$\mathscr{B}(\varrho, \mathbf{u}, \mathbf{w}) = \varrho(\mathbf{N}^{*})^{-1}\Big(\mathbf{u}\,\nabla(\mathbf{N}^{-1}\mathbf{w})\Big).$$

Sensitivity analysis

The specific structure of the matrix **N** does not play any particular role in the analysis. Therefore, we consider a general problem of the existence, uniqueness and dependence on coefficients of the solutions to compressible NSE under the assumption that **N** is a given matrix-valued function which is close, in an appropriate norm, to the identity mapping **I** and coincides with **I** in the vicinity of Σ. We consider the perturbations of the obstacle S only. We write $\mathbf{u}$ and ϱ for $\mathbf{u}_\varepsilon$ and ϱ_ε, when studying the well-posedness and dependence on **N**. Before formulation of main results we write the governing equation in more transparent form using the change of unknown functions proposed by Padula.

We introduce *the effective viscous pressure*

$$q = \frac{R}{\delta} p(\varrho) - \lambda \mathfrak{g}^{-1} \operatorname{div} \mathbf{u},$$

and rewrite equations in the equivalent form

$$\Delta \mathbf{u} - \nabla q = \mathscr{A}(\mathbf{u}) + R\mathscr{B}(\varrho, \mathbf{u}, \mathbf{u}) \text{ in } \Omega, \tag{4a}$$

$$\operatorname{div} \mathbf{u} = a\sigma_0 p(\varrho) - \frac{\mathfrak{g}q}{\lambda} \text{ in } \Omega, \tag{4b}$$

$$\mathbf{u} \cdot \nabla \varrho + \mathfrak{g}\sigma_0 p(\varrho)\, \varrho = \frac{\mathfrak{g}q}{\lambda} \varrho \text{ in } \Omega, \tag{4c}$$

$$\mathbf{u} = \mathbf{U} \text{ on } \Sigma, \quad \mathbf{u} = 0 \text{ on } \partial S, \tag{4d}$$

$$\varrho = \varrho_0 \text{ on } \Sigma_{\text{in}}. \tag{4e}$$

where $\sigma_0 = R/(\lambda\delta)$.

In the new variables $(\mathbf{u}, q, \varrho)$ the expression for the force $\mathbf{J}$ reads

$$\mathbf{J} = -\int_\Omega \left[\mathfrak{g}^{-1}\big(\mathbf{N}^*\nabla(\mathbf{N}\mathbf{u}) + \nabla(\mathbf{N}\mathbf{u})^*\mathbf{N} - \operatorname{div}\mathbf{u}\big) - q - R\,\varrho\mathbf{u}\otimes\mathbf{u}\right]\mathbf{N}^*\nabla\eta\, dx.$$

where $\eta \in C^\infty(\Omega)$ is an arbitrary function, which is equal to 1 in an open neighborhood of the obstacle S and 0 in a vicinity of Σ. The value of $\mathbf{J}$ is independent of the choice of the function η. We assume that $\lambda \gg 1$ and $R \ll 1$, which corresponds to almost incompressible flow with low Reynolds number.

In such a case, the *approximate solutions* to compressible Navier-Stokes equations can be chosen in the form $(\varrho_0, \mathbf{u}_0, q_0)$, where ϱ_0 is a constant in the boundary conditions, and $(\mathbf{u}_0, q_0)$ is a solution to the Stokes equations,

$$\Delta \mathbf{u}_0 - \nabla q_0 = 0, \quad \operatorname{div} \mathbf{u}_0 = 0 \text{ in } \Omega, \tag{5}$$
$$\mathbf{u}_0 = \mathbf{U} \text{ on } \Sigma, \quad \mathbf{u}_0 = 0 \text{ on } \partial S, \quad \Pi q_0 = q_0 \, .$$

In our notations Π is the projector,

$$\Pi u = u - \frac{1}{\text{meas}\,\Omega} \int_\Omega u \, dx .$$

Equations (5) can be obtained as the limit of equations (4) for the passage $\lambda \to \infty$, $R \to 0$.

It follows from the standard elliptic theory that for the boundary $\partial\Omega \in C^\infty$, we have $(\mathbf{u}_0, q_0) \in C^\infty(\Omega)$. We look for solutions to problem (4) in the form

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{v}, \quad \varrho = \varrho_0 + \varphi, \quad q = q_0 + \lambda\sigma_0 p(\varrho_0) + \pi + \lambda m, \quad (6)$$

with the unknowns functions $\vartheta = (\mathbf{v}, \pi, \varphi)$ and the unknown constant m. Substituting (6) into (4) we obtain the following boundary problem for ϑ,

$$\Delta\mathbf{v} - \nabla\pi = \mathscr{A}(\mathbf{u}) + R\mathscr{B}(\varrho, \mathbf{u}, \mathbf{u}) \text{ in } \Omega,$$

$$\operatorname{div}\mathbf{v} = \mathfrak{g}\Big(\frac{\sigma}{\varrho_0}\varphi - \Psi[\vartheta] - m\Big) \text{ in } \Omega,$$

$$\mathbf{u}\cdot\nabla\varphi + \sigma\varphi = \Psi_1[\vartheta] + m\mathfrak{g}\varrho \text{ in } \Omega,$$

$$\mathbf{v} = 0 \text{ on } \partial\Omega, \quad \varphi = 0 \text{ on } \Sigma_{\text{in}}, \quad \Pi\pi = \pi,$$

where

$$\Psi_1[\vartheta] = \mathfrak{g}\Big(\varrho\Psi[\vartheta] - \frac{\sigma}{\varrho_0}\varphi^2\Big) + \sigma\varphi(1-\mathfrak{g}),$$

$$\Psi[\vartheta] = \frac{q_0 + \pi}{\lambda} - \frac{\sigma}{p'(\varrho_0)\varrho_0}H(\varphi),$$

$$\sigma = \sigma_0 p'(\varrho_0)\varrho_0, \quad H(\varphi) = p(\varrho_0+\varphi) - p(\varrho_0) - p\prime(\varrho_0)\varphi,$$

the vector field **u** and the function ϱ are given by (6).
Finally, we specify the constant m. In our framework, in contrast to the case of homogeneous boundary problem, the solution to such a problem is not trivial.

Note that, since $\operatorname{div} \mathbf{v}$ is of the null mean value, the right-hand side of equation

$$\mathbf{u} \cdot \nabla \varphi + \sigma \varphi = \Psi_1[\vartheta] + m \mathfrak{g} \varrho \ \text{ in } \ \Omega,$$

must satisfy the compatibility condition

$$m \int_\Omega \mathfrak{g} \, dx = \int_\Omega \mathfrak{g} \big(\frac{\sigma}{\varrho_0} \varphi - \Psi[\vartheta] \big) \, dx,$$

which formally determines m. This choice of m leads to essential mathematical difficulties.

Hence, the question of solvability of the linearized equations can be reduced to the question of solvability of the boundary value problem for nonlocal transport equation

$$\mathbf{u}\nabla\varphi + \sigma\Pi\varphi = f \ ,$$

which is very difficult because of the loss of maximum principle. In fact, this question is concerned with the problem of the control of the total gas mass in compressible flows. Recall that the absence of the mass control is the main obstacle for proving the global solvability of inhomogeneous boundary problems for compressible Navier-Stokes equations, we refer to the monograph by P.-L. Lions for discussion. In order to cope with this difficulty we write the compatibility condition in a sophisticated form, which allows us to control the total mass of the gas.

Compability conditions

To this end we introduce the auxiliary function ζ satisfying the equations

$$-\operatorname{div}(\mathbf{u}\zeta) + \sigma\zeta = \sigma\mathfrak{g} \ \text{ in } \ \Omega, \quad \zeta = 0 \ \text{ on } \ \Sigma_{\text{out}}, \tag{7}$$

and fix the constant m as follows

$$m = \varkappa \int_\Omega (\varrho_0^{-1}\Psi_1[\vartheta]\zeta - \mathfrak{g}\Psi[\vartheta])\, dx, \quad \varkappa = \Big(\int_\Omega \mathfrak{g}(1-\zeta-\varrho_0^{-1}\zeta\varphi)\, dx \Big)^{-1}. \tag{8}$$

In this way the auxiliary function ζ becomes an integral part of the solution to the boundary value problem.

Now, our aim is to prove the existence and uniqueness of solutions to the boundary value problem and investigate the dependence of the solutions on matrices **N**.

The existence and uniqueness of small solutions is shown under the following geometrical conditions.

Geometrical conditions on the flow region. We assume that a surface $\Sigma = \Sigma_{\text{in}} \cup \Sigma_{\text{out}} \cup \Gamma$ and a given vector field **U** satisfy the following conditions, referred to as the **emergent vector field condition**.

Remark *The important particular case, e.g., for the numerical analysis, is the case of the hold-all domain strictly convex, and of the constant boundary vector field for the velocity, in such a case the* **emergent vector field condition** *is satisfied and the geometrical part of the drag shape gradient is null.*

Emergent field condition

In the general case, the condition on the solvability of the first boundary value problem in a bounded domain can be given as follows.

The set Γ is a a closed C^∞ one-dimensional manifold. Moreover, there is a positive constant c such that

$$\mathbf{U} \cdot \nabla(\mathbf{U} \cdot \mathbf{n}) > c > 0 \ \ \textit{on} \ \ \Gamma. \tag{9}$$

Since the vector field $\mathbf{U}$ is tangent to $\partial\Omega$ on Γ, the quantity in the left-hand side of (9) is well defined.

This condition has simple geometric interpretation, that $\mathbf{U} \cdot \mathbf{n}$ only vanishes up to the first order at Γ, and for each point $P \in \Gamma$, the vector $\mathbf{U}(P)$ points to the part of $\partial\Omega$ where $\mathbf{U}$ is an exterior vector field.

Small perturbations of approximate solutions

The existence of small perturbations of approximate solutions is proved by the application of the Schauder fixed point theorem. Next step, is the proof of stability of such solutions with respect to the perturbations of the differential operators by the matrix function **N** which is closed to the identity matrix. The result obtained furnishes the so-called material derivatives of the solutions to the compressible Navier-Stokes equations as well as to the shape differentiability of the drag functional.

Material derivatives

We have the boundary value problem for compressible Navier Stokes equations in fixed domain, with coefficients depending on the matrix **N**, it means on the small parameter ε. Thus, we are in position to evaluate the material derivatives of solutions and, as a result, the shape gradient of the drag functional.

Finite differences

Recall that

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{v}, \quad \varrho = \varrho_0 + \varphi, \quad q = q_0 + \lambda\sigma_0 p(\varrho_0) + \pi + \lambda m, \quad (10)$$

with the unknowns functions $\vartheta = (\mathbf{v}, \pi, \varphi)$ and the unknown constant m. Denote the solution for $\varepsilon = 0$ by (ϑ, m, ζ), and define the finite differences with respect to ε

$$(\mathbf{w}_\varepsilon, \omega_\varepsilon, \psi_\varepsilon) = \varepsilon^{-1}(\vartheta - \vartheta(\varepsilon)),$$
$$\xi_\varepsilon = \varepsilon^{-1}(\zeta - \zeta(\varepsilon)),$$
$$n_\varepsilon = \varepsilon^{-1}(m - m(\varepsilon)).$$

The limit $(\mathbf{w}, \omega, \psi, \xi, n) = \lim_{\varepsilon\to 0}(\mathbf{w}_\varepsilon, \omega_\varepsilon, \psi_\varepsilon, \xi_\varepsilon, n_\varepsilon)$ is a solution to linearized equations

Linearized equations

$$
\begin{aligned}
\Delta \mathbf{w} - \nabla \omega &= R\, \mathscr{C}_0(\mathbf{w}, \psi) + \mathscr{D}_0(\mathbf{D}) \ \text{ in } \ \Omega, \\
\operatorname{div} \mathbf{w} &= b^0_{21}\, \psi - b^0_{22}\, \omega + b^0_{23}\, n + b^0_{30}\, \mathfrak{d} \ \text{ in } \ \Omega, \\
\mathbf{u} \nabla \psi + \sigma \psi &= -\mathbf{w} \cdot \nabla \varphi + b^0_{11}\, \psi + b^0_{12}\, \omega + b^0_{13}\, n + b^0_{10}\, \mathfrak{d} \ \text{ in } \ \Omega, \\
-\operatorname{div}(\mathbf{u}\xi) + \sigma \xi &= \operatorname{div}(\zeta \mathbf{w}) + \sigma \mathfrak{d} \ \text{ in } \ \Omega, \\
\mathbf{w} = 0 \ \text{ on } \ \partial\Omega, \quad \psi = 0 \ \text{ on } \ \Sigma_{\mathrm{in}}, & \qquad \xi = 0 \ \text{ on } \ \Sigma_{\mathrm{out}}, \\
\omega - \Pi \omega = 0, \quad n = \varkappa \int_\Omega \big(b^0_{31}\, \psi + b^0_{32}\, \omega &+ b^0_{34}\, \xi + b^0_{30}\, \mathfrak{d}\big)\, dx,
\end{aligned} \tag{11}
$$

where $\mathfrak{d} = 1/2 \, \mathrm{Tr}\, \mathbf{D}$, the variable coefficients b^0_{ij} and the operators $\mathscr{C}_0$, $\mathscr{D}_0$, are defined by the appropriate formulae.

Material derivatives

Theorem 1 There exist the limits,

$$\mathbf{w}_\varepsilon \to \mathbf{w} \text{ weakly in } \mathcal{H}_0^{1-s,r'}(\Omega), \quad n_\varepsilon \to n \text{ in } \mathbb{R},$$
$$\psi_\varepsilon \to \psi, \quad \omega_\varepsilon \to \omega, \quad \xi_\varepsilon \to \xi \text{ } (*)\text{-weakly in } \mathbb{H}^{-s,r'}(\Omega) \text{ as } \varepsilon \to 0, \tag{12}$$

where the material derivatives, vector field $\mathbf{w}$, functionals ψ, ω, ξ, and the constant n are given by the weak solution to linearized problem (11).

Shape derivative of the drag functional

Theorem 2 There exists the shape derivative of the drag functional in the following form

$$\frac{d}{d\varepsilon} J_D(S_\varepsilon)\Big|_{\varepsilon=0} = L_e(\mathbf{T}) + L_u(\mathbf{w}, \omega, \psi).$$

Notation

The linear forms L_e and L_u are required in order to define the shape derivative of the drag functional. The first form L_e is called the geometrical part of the shape gradient. It depends on the transformation T and the solution to the state equation. We can show that for the strictly convex hold-all domain and for the constant vector fields on the boundary, the form is null.
The second form L_u depends on the material derivatives of solutions to the state equation. The adjoint state is introduced in order to eliminate the material derivatives and derive the standard form of the shape derivative of the drag functional.

$$L_e(\mathbf{T}) = \int_\Omega \operatorname{div}\mathbf{T}(\nabla\mathbf{u} + \nabla\mathbf{u}^* - \operatorname{div}\mathbf{u}\mathbf{I})\mathbf{U}_\infty\, dx -$$
$$\int_\Omega \left[\nabla\mathbf{u} + \nabla\mathbf{u}^* - \operatorname{div}\mathbf{u} - q\mathbf{I} - R\varrho\mathbf{u}\otimes\mathbf{u}\right]\mathbf{D}\nabla\eta\cdot\mathbf{U}_\infty\, dx -$$
$$\int_\Omega \left[\mathbf{D}^*\nabla\mathbf{u} + \nabla\mathbf{u}^*\mathbf{D} + \nabla(\mathbf{D}\mathbf{u}) + \nabla(\mathbf{D}\mathbf{u})^*\right]\nabla\eta\cdot\mathbf{U}_\infty\, dx$$

and

$$L_u(\mathbf{w},\omega,\psi) = \int_\Omega \mathbf{w}\left[\Delta\eta\mathbf{U}_\infty + R\varrho(\mathbf{u}\cdot\nabla\eta)\mathbf{U}_\infty + R\varrho(\mathbf{u}\cdot\mathbf{U}_\infty)\nabla\eta\right] dx$$
$$+\langle\omega, \nabla\eta\cdot\mathbf{U}_\infty\rangle + R\langle\psi, (\mathbf{u}\cdot\nabla\eta)(\mathbf{u}\cdot\mathbf{U}_\infty)\rangle.$$

Singular limits of volume integrals

Assume that the vector field

$$\mathbf{T}(x) = f(x)\mathbf{n}(x) \;\text{ for }\; x \in \partial S.$$

is extended to a small neighbourhood of ∂S, the parameter of the extension is τ.

Lemma

For any $\psi \in C(\Omega)$,

$$\begin{aligned} \lim_{\tau \searrow 0} \int_{\Omega} \psi(x)\mathbf{T}'_{\tau}(x)\,dx &= \int_{S} \psi(x) f(x)\mathbf{n} \otimes \mathbf{n}\,ds, \\ \lim_{\tau \searrow 0} \int_{\Omega} \psi(x)\mathrm{Tr}\;\mathbf{T}'_{\tau}(x)\,dx &= \int_{S} \psi(x) f(x)\,ds \end{aligned} \tag{13}$$

Boundary shape gradient

$$\mathbf{T}(x) = f(x)\mathbf{n}(x) \text{ for } x \in \partial S.$$

It is reasonable to eliminate η and $\mathbf{T}$ from formulae for the shape gradient and the adjoint state equations and reformulate the expression for forms L_e and L_u in terms of the normal shift $f(\cdot)$ only.

Theorem 3 If the perturbed surface ∂S_ε is defined by the extended mapping $\mathbf{I} + \varepsilon\mathbf{T}$ with $f \in C^\infty(\partial S)$, then

$$L_e(\mathbf{T}) = 0, \tag{14}$$

and

$$L_u(\mathbf{w}, \psi, \omega) = \int_{\partial S} f(x)\big[\big(b_{10}\varsigma + b_{20}^0 g + \sigma\upsilon + \varkappa b_{30}\big) + (\partial_n \mathbf{h}\cdot\mathbf{n})(\partial_n \mathbf{u}\cdot\mathbf{n})\big] \tag{15}$$

with the adjoint state variables $(\mathbf{h}, g, \varsigma, \upsilon)$ satisfying the following equations and boundary conditions

$$(\mathbf{h}, \varpi, \varsigma, \upsilon, I) = (\mathbf{H}, G, Z, 0, 0) + (\tilde{\mathbf{h}}, \tilde{\varpi}, \tilde{\varsigma}, \tilde{\upsilon}, \tilde{I}), \tag{16}$$

$$\Delta \mathbf{H} - \nabla G - R\varrho\,(\mathbf{H}\nabla \mathbf{u} + \mathbf{u}\nabla \mathbf{H}) = 0, \tag{17a}$$

$$\operatorname{div} \mathbf{H} + \lambda^{-1}\Pi G = 0, \tag{17b}$$

$$-\operatorname{div}(\mathbf{u}Z) + \sigma Z = R(\mathbf{u}\nabla \mathbf{u}) \cdot \mathbf{H} + b_{21}G + b_{11}Z, \tag{17c}$$

$$\mathbf{H} = 0 \ \text{ on } \ \Sigma, \quad \mathbf{H} = -\mathbf{U} \ \text{ on } \ \partial S, \quad Z = 0 \ \text{ on } \ \Sigma_{\text{out}}. \tag{17d}$$

$$\Delta \tilde{\mathbf{h}} - \nabla \tilde{\varpi} - R\varrho\,(\tilde{\mathbf{h}}\nabla \mathbf{u} + \mathbf{u}\nabla \tilde{\mathbf{h}}) + (Z + \tilde{\varsigma})\nabla\varphi + \zeta\nabla\tilde{\upsilon} = 0 \tag{18a}$$

$$\operatorname{div} \tilde{\mathbf{h}} - \Pi(b_{12}(Z + \tilde{\varsigma}) + \lambda^{-1}\Pi\tilde{\varpi} = 0 \tag{18b}$$

$$-\operatorname{div}(\mathbf{u}\tilde{\varsigma}) + \sigma\tilde{\varsigma} = R(\mathbf{u}\nabla\mathbf{u}) \cdot (\tilde{\mathbf{h}}) + b_{21}\tilde{\varpi} + b_{11}\tilde{\varsigma} + \varkappa b_{13}\tilde{l} \quad \text{(18c)}$$

$$\mathbf{u}\nabla\tilde{\upsilon} + \sigma\tilde{\upsilon} - \kappa b_{34}\tilde{l} = 0 \quad \text{(18d)}$$

$$\int_{\Omega} b_{13}(Z + \tilde{\varsigma})\, dx + \tilde{l} = 0 \quad \text{(18e)}$$

$$\tilde{\mathbf{h}} = 0 \ \text{ on } \ \partial\Omega, \quad \tilde{\varsigma} = 0 \ \text{ on } \ \Sigma_{\text{out}}, \quad \tilde{\upsilon} = 0 \ \text{ on } \ \Sigma_{\text{in}}. \quad \text{(18f)}$$

In addition, there exists σ_0 with the following properties. For any $\sigma > \sigma_0$, there are λ_0 and R_0, depending only on σ, ∂S and $\mathbf{U}$, so that for every $R < R_0$ and $\lambda > \lambda_0$ the adjoint state equations have a solution $(\mathbf{h}, \varpi, \varsigma, \upsilon, l) \in W^{2,2}(\Omega) \times (W^{1,2}(\Omega))^2 \times \mathbb{R}$.
The boundary shape gradient is independent on the function η and depends only on the restriction of the transformation field to the boundary.
The proof is based on the singular limits of volume integrals if the supports of the transformation fields converge to the boundary ∂S.

Numerical example in two spatial dimensions; Oberwolfach (2009), MMAR2009

Recall that in our framework the hydro-dynamical force acting on the body S is defined by the formula,

$$\mathbf{J}(S) = -\int_{\partial S} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T + (\lambda - 1)\mathrm{div}\mathbf{u} I - \frac{R}{\delta} p I)\mathbf{n}\, dS.$$

In a frame attached to the moving body the drag is the component of **J** parallel to $\mathbf{U}_\infty$,

$$J_D(S) = \mathbf{U}_\infty \cdot \mathbf{J}(S), \tag{19}$$

and the lift is the component of **J** in the direction orthogonal to $\mathbf{U}_\infty$. For the fixed data, the drag can be regarded as a functional depending on the shape of the obstacle S.

The system of compressible Navier-Stokes equations transforms to

$$\Delta \mathbf{u} - \nabla q = R\varrho \nabla \mathbf{u} \cdot \mathbf{u}$$
$$\mathrm{div}\,(\mathbf{u}) = \sigma_0 p(\varrho) - \frac{1}{\lambda} q \qquad (20)$$
$$\mathbf{u}^T \nabla \varrho + \sigma_0 p(\varrho)\varrho - \frac{1}{\lambda} q \varrho = 0$$

where $\sigma_0 = R/(\lambda\delta)$, $\sigma = \sigma_0 p_0 \gamma$.
Here, the effective viscous pressure is used

$$q = \frac{R}{\delta} p(\varrho) - \lambda \mathrm{div}\,(\varrho \mathbf{u}) \qquad (21)$$

In addition, if we introduce a smooth function η defined in Ω and satisfying boundary conditions $\eta = 1$ on ∂S, $\eta = 0$ on Σ, then the expression for drag takes on the form

$$J_D(S) = -\mathbf{U}_\infty \cdot \int_\Omega \Big(\nabla \mathbf{u} + \nabla \mathbf{u}^T - \mathrm{div}\,(\mathbf{u}) I \\ - qI - R\varrho \mathbf{u} \otimes \mathbf{u}\Big) \nabla \eta \, dx. \tag{22}$$

This can be verified by integration by parts. In our case we take η harmonic in Ω.
Assuming $\lambda \gg 1$ and $R \ll 1$ (weakly compressible flow) we may approximate solution to (20) by means of the small perturbation with respect to the solution of the Stokes problem

$$\begin{aligned} \Delta \mathbf{u}_0 - \nabla q_0 = 0 \\ \mathrm{div}(\mathbf{u}_0) = 0 \\ \mathbf{u}_0 = \mathbf{U} \text{ on } \Sigma, \quad \mathbf{u}_0 = 0 \text{ on } \partial S, \; M(q_0) = 0 \end{aligned} \tag{23}$$

where $M(\cdot)$ denotes mean value on Ω. We assume these perturbations in the form

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{v}, \qquad \varrho = \varrho_0 + \phi \tag{24}$$

$$q = q_0 + \lambda\sigma_0 p_0 + \pi + m \tag{25}$$

where $\mathbf{v}, \ \phi, \ \pi$ are unknown functions and m the unknown constant. Taking into account (23) the system of equations for $\mathbf{v}, \ \phi, \ \pi$ is

$$\begin{aligned}
\Delta\mathbf{v} - \nabla\pi &= R\varrho\nabla\mathbf{u}\cdot\mathbf{u} \\
\mathrm{div}(\mathbf{v}) &= \sigma_0 p(\varrho) - \frac{1}{\lambda}q \\
\mathbf{u}^T\nabla\phi + \sigma\phi &= \frac{1}{\lambda}\varrho(q_0 + \pi) \\
&\quad - \sigma_0\varrho(p - p_0) + \sigma_0 p_0 p'(\varrho_0)(\varrho - \varrho_0)
\end{aligned} \tag{26}$$

with boundary conditions

$$\mathbf{v} = 0 \ \text{ on } \ \partial\Omega, \quad \phi = 0 \ \text{ on } \ \Sigma_{in}, \quad M(\pi) = 0$$

and the condition $M(\mathrm{div}(\mathbf{v})) = 0$, which translates to

$$m = \frac{\sigma_0}{|\Omega|} \int_\Omega \left[p(\varrho) - p(\varrho_0)\right] dx. \tag{27}$$

It is convenient to introduce an additional equation

$$\begin{aligned} -\,\mathrm{div}(\mathbf{u}\zeta) + \sigma\zeta = \sigma \ \text{ in } \ \Omega \\ \zeta = 0 \ \text{ on } \ \Sigma_{out} \end{aligned} \tag{28}$$

and express m

$$m = \varkappa \int_\Omega (\varrho_0^{-1}\Psi_1[\vartheta]\zeta - \mathfrak{g}\Psi[\vartheta])\, dx, \tag{29}$$

$$\varkappa = \Big(\int_\Omega \mathfrak{g}(1 - \zeta - \varrho_0^{-1}\zeta\varphi)\, dx \Big)^{-1}. \tag{30}$$

In order to introduce the perturbation of the obstacle we introduce the transformation of the domain Ω by means of the mapping

$$\mathcal{T}(\mathbf{x}) = \mathbf{x} + \varepsilon \mathbf{T}(\mathbf{x}) \tag{31}$$

where $\mathbf{T}(\mathbf{x}) = 0$ on Σ and $\mathbf{T}_{|\partial S}$ describes the movement of the boundary of S. We assumed $\mathbf{T} = [t_1, t_2]^T$ in the particular form, where t_i satisfy equations

$$\Delta t_i = 0 \text{ in } \Omega, \quad t_i = 0 \text{ on } \Sigma \tag{32}$$
$$t_i = h_i \text{ on } \partial S, \quad i = 1, 2.$$

Here $h_i(\mathbf{x})$ represent the shift of the point $\mathbf{x}$ on ∂S. In the sequel we denote the solutions of the same equations (26),(27),(28) in the transformed domain $\Omega_\varepsilon = \mathcal{T}(\Omega)$ by $\mathbf{v}(\varepsilon)$, $\phi(\varepsilon)$, $\pi(\varepsilon)$, $m(\varepsilon)$, $\zeta(\varepsilon)$. By means of the inverse transformation $\mathcal{T}^{-1}$ all these functions may be shifted again to the unperturbed domain Ω, together with defining equations. Therefore we may consider them as functions defined on Ω and formally compute derivatives

$$\begin{aligned}
\mathbf{w} &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\mathbf{v} - \mathbf{v}(\varepsilon)] \\
\omega &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\pi - \pi(\varepsilon)] \\
\xi &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\zeta - \zeta(\varepsilon)] \\
\psi &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\phi - \phi(\varepsilon)] \\
n &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[m - m(\varepsilon)]
\end{aligned} \tag{33}$$

Let us denote by $\mathcal{P}$ the set of solutions in the unperturbed domain, $\mathcal{P} = [\mathbf{v}, \phi, \pi, m, \zeta]$. It can be shown, that the above defined derivatives satisfy in Ω the following system of equations

$$\begin{aligned} \Delta \mathbf{w} - \nabla \omega &= \mathbf{F}_1(\mathcal{P}, \mathbf{w}, \psi, D) \\ \operatorname{div}(\mathbf{w}) &= F_2(\mathcal{P}, \psi, n, \omega, D) \\ \mathbf{u}^T \nabla \psi + \sigma \psi &= F_3(\mathcal{P}, \psi, n, \omega, D) \\ -\operatorname{div}(\mathbf{u}\xi) + \sigma \xi &= F_4(\mathcal{P}, \omega, D) \end{aligned} \tag{34}$$

with boundary conditions

$$\mathbf{w} = 0 \;\text{ on }\; \partial\Omega, \quad \psi = 0 \;\text{ on }\; \Sigma_{in}, \quad \xi = 0 \;\text{ on }\; \Sigma_{out}$$

as well $M(\omega) = 0$ and

$$n = \int_\Omega F_5(\psi, \omega, \xi, D)\, dx .$$

The matrix D characterizing the transformation is given by

$$D = \operatorname{div}(\mathbf{T})I - \nabla \mathbf{T}.$$

The functions $\mathbf{F}_1, F_2, F_3, F_4, F_5$ are complicated expressions in terms its arguments. For illustration we show only $\mathbf{F}_1$:

$$\begin{aligned}\mathbf{F}_1(\mathcal{P}, \mathbf{w}, \psi, D) &= R^2\big(\phi\mathbf{u}\nabla\mathbf{u} + \varrho\mathbf{w}\nabla\mathbf{u} + \varrho\mathbf{u}\nabla\mathbf{w}\big) \\ &+ R\mathbf{u}\nabla(D\mathbf{u}) + RD^T(\mathbf{u}\nabla\mathbf{u}) \\ &+ \mathrm{div}\Big[(D + D^T)\nabla\mathbf{u} - \frac{1}{2}Tr(D)\nabla\mathbf{u}\Big] \\ &- D\Delta\mathbf{u} - \Delta(D\mathbf{u})\end{aligned}$$

It may be proved that using these functions the expression for the shape derivative of the drag takes on the form

$$\frac{d}{d\varepsilon}J_D(S_\varepsilon)_{|\varepsilon=0} = L_1 + L_2 + L_3 + L_4 + L_5 \tag{35}$$

where

$$L_1 = \int_\Omega \mathrm{div}(\mathbf{T})(\nabla\mathbf{u} + \nabla\mathbf{u}^T - \mathrm{div}(\mathbf{u})I)\nabla\eta \cdot \mathbf{U}_\infty\, dx$$

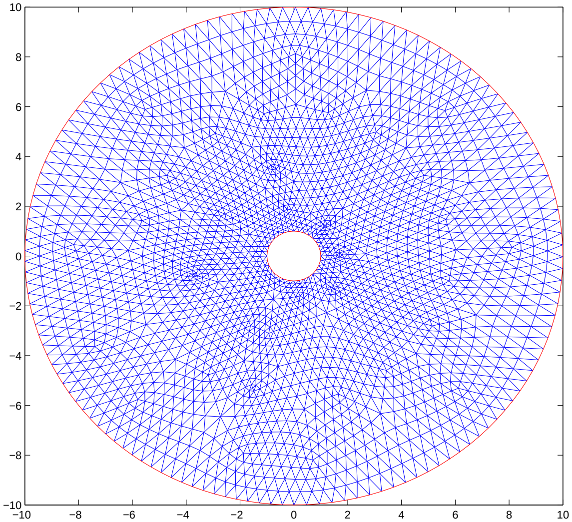
$$L_2 = -\int_\Omega \big(\nabla \mathbf{u} + \nabla \mathbf{u}^T - \mathrm{div}(\mathbf{u})I - qI - R\varrho \mathbf{u} \otimes \mathbf{u}\big) D^T \nabla \eta \cdot \mathbf{U}_\infty \, dx$$

$$L_3 = -\int_\Omega \big(D^T \nabla \mathbf{u} + \nabla \mathbf{u}^T D + \nabla(D\mathbf{u}) + \nabla(D\mathbf{u})^T\big) \nabla \eta \cdot \mathbf{U}_\infty \, dx$$

$$L_4 = \int_\Omega \mathbf{w} \cdot \big(\Delta \eta \mathbf{U}_\infty + R\varrho(\mathbf{u} \cdot \nabla \eta)\mathbf{U}_\infty + R\varrho(\mathbf{u} \cdot \mathbf{U}_\infty)\nabla \eta\big) \, dx$$

$$L_5 = \int_\Omega \big[\omega \nabla \eta \cdot \mathbf{U}_\infty + \psi(\mathbf{u} \cdot \nabla \eta)(\mathbf{u} \cdot \mathbf{U}_\infty)\big] \, dx$$

It can be shown that under reasonable regularity assumptions $\mathbf{v} \in [C^1(\Omega)]^3$ and $\pi, \phi, \zeta \in C(\Omega)$. However, the convergence of limits in (33) takes place in very weak spaces, see above. The preliminary numerical computations were performed in $\mathbb{R}^2$. The domain B constituted a ball $B = B(0, R)$ and the initial obstacle was $S = B(0, r)$ with $R/r = 10$. The domain $\Omega = B \setminus S$ was triangulated (see Fig.1) For solving the Stokes Problem (23) the flow velocity $\mathbf{u}_0$ was approximated by piecewise P_1 (first order polynomial) functions on triangles, while for q_0 piecewise P_0 (constant) functions were used. For regularization of the pressure q_0 the penalty term containing interelement jumps was applied. The same elements were used for approximating $\mathbf{v}, \pi$. The functions ϕ, ζ were approximated by P_1 elements.


10
8
6
4
2
0
−2
−4
−6
−8
−10
−10 −8 −6 −4 −2 0 2 4 6 8 10

However, the system (26) is nonlinear. Therefore it was solved iteratively, using Ishikawa [2] fixed point procedure. The right–hand sides were taken as functions of $\mathcal{P}$, denoted by $\mathcal{R}(\mathcal{P})$. As a result (26) takes on the form

$$\mathcal{P} = \mathcal{S}^{-1}[\mathcal{R}(\mathcal{P})]$$

where $\mathcal{S}^{-1}$ represents solving the system with given $\mathcal{R}$. This justifies using fixed point method. The Ishikawa algorithm for finding x such that $x = \Phi(x)$ may be written as the following iteration:

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n \Phi(x_n) \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n \Phi(y_n) \end{aligned}$$

where $0 \leq \alpha_n, \beta_n < 1$,

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$$

and

$$\sum_{n=0}^{\infty} \alpha_n = \infty.$$

In our case it was taken $\alpha_n = \beta_n = 1/\sqrt{n+1}$.
For the range of flow parameters used in computations the convergence was quite quick. The same procedure and approximation was used for solving the system (34), since it has the same structure. It was convenient, because even if $\mathbf{w}, \omega, \xi, \psi$ enter the right-hand side linearly, the expression for n makes iterations necessary. In the weak formulation the second derivatives of $\mathbf{u}$ disappear and the particular form of D (t_i harmonic in Ω) could be exploited.
As it is easily seen, the shape derivative of the drag (35) is computed for the particular transformation field $\mathbf{T}$ and resulting matrix D. The general movement of the curve ∂S was expressed as linear combination of "bump" deformations, which

were constructed in the following way.
First, the boundary ∂S was approximated by the closed, smooth (C^2) spline passing through all the discretization nodes on ∂S and parametrized by arclength s

$$\gamma = \gamma(s), \quad s \in [0, L], \quad \gamma(s_k) = \mathbf{p}_k, \quad k = 1 \ldots, K.$$

Next at each point $\mathbf{p}_k = \gamma(s_k)$ the outer normal vector was computed

$$\mathbf{n}_k = \frac{N\gamma'(s_k)}{\|\gamma'(s_k)\|}, \qquad N = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

which indicated the direction of movement for this point. Finally, the "bump" function was defined

$$b_k(s) = \exp\Big[-\Big(\frac{dist(s, s_k)}{d_0}\Big)^2\Big],$$

where

$$dist(s, s_k) = \min[|s - s_k|, L - |s - s_k|]$$

is the minimal distance from s to s_k (remember that γ is closed) and d_0 represents the width of the "bump". Using this function and taking $\mathbf{h}(\mathbf{p}_j) = b_k(s_j)\mathbf{n}_j$, $j = 1, \ldots, K$ one can compute the corresponding $\mathbf{T}_k = \mathbf{T}(\mathbf{h})$ and D_k.

Having $D := D_k$ it was possible to solve the system (34) and obtain the shape derivative (35). This procedure had to be repeated K times, for each vertex on ∂S.

After performing small movements of boundary points along $\mathbf{n}_k$ it has been observed that in the regions of bigger curvature the points $\mathbf{p}_k$ tended to converge to each other, causing even the overlap of triangles after several iteration steps. To remedy this behavior the following procedure was used after each step.

Taking new positions of points $\mathbf{p}'_k$ as nodes the new spline $\gamma_1(s)$ was computed, $\gamma_1(s'_k) = \mathbf{p}'_k$. Then the parameters s'_k

were slightly shifted, so that the distances between neighboring points along γ_1 were equal on all the new boundary, i.e. the new nodes were uniformly distributed. This prevented spoiling the quality of triangulation.

In numerical computations we considered the problem of drag minimization and, for illustration purposes only, drag maximization. We describe briefly the numerical results given in Figures 2-6. The results are only preliminary, since they are obtained with few steps of the simple gradient method, with the shape gradient numerically evaluated according to the formula given in (35). Triangulation and computational domain is shown in Fig.1. The flow is from the left, Reynolds number is $R = 0.01$, viscosity ratio $\lambda = 100$, the flow velocity is $U_1 = 1, U_2 = 0$ on outer boundary. The coefficient in gas law is $\gamma = 5/3$. In order to prevent moving the obstacle toward the boundary of the computational region, it is assumed that its gravity center is fixed at the origin. The total volume of the

obstacle is kept constant.
The optimized shapes after few iterations are shown. The computations in case of drag minimization seem to converge to some shape, in case of drag maximization the situation is different, because the optimal shape cannot exist. The results shown are raw, in the sense that there was no attempt to exploit the symmetry of the problem. In view of this remark they look satisfactory.

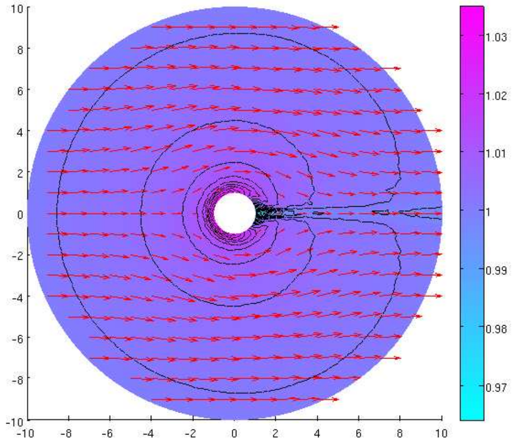


Figure: Initial flow **u** and pressure p.

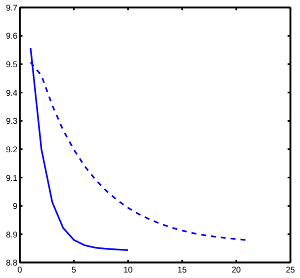


Figure: Shape of minimal drag for rough (dashed line) and finer discretizations. On the left history of optimization.

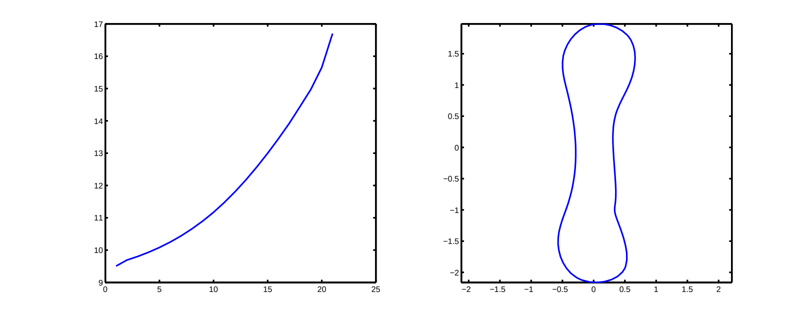


Figure: Shape after few steps of drag maximization and the history of drag values.

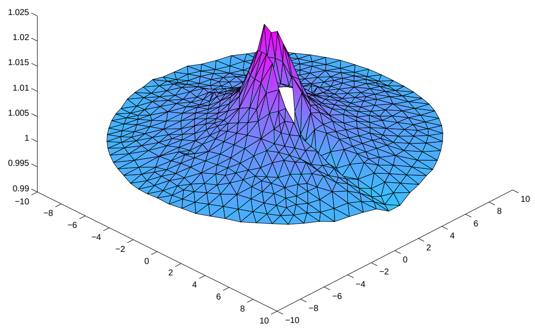


Figure: Pressure distribution around shapes of minimal drag.

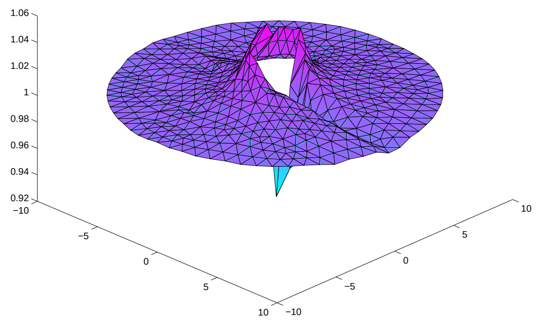


Figure: Pressure distribution around shapes of maximal drag.

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Conclusions

The drag functional for compressible Navier Stokes equations is shape differentiable. Therefore, the numerical methods of shape optimization can be applied in order to solve such optimal design problems like minimization of the drag and/or maximization of the lift.
The same result can be obtained for the complete system with the equation for the temperature, this is the subject of the current studies since even the existence of the solutions is an open problem.