Efficient Optimal Control of Young Concrete Mechanical Properties

Olaf Benedix, Dominik Meidner, Boris Vexler
Zentrum Mathematik, M1, TU München

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Outline

controls

cast mixture
q_1 cement
q_2 fly ash
q_3 water
q_4 aggregate

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initial temperature
$y(0) = q_5$

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stripping point
$t_0 = q_6$

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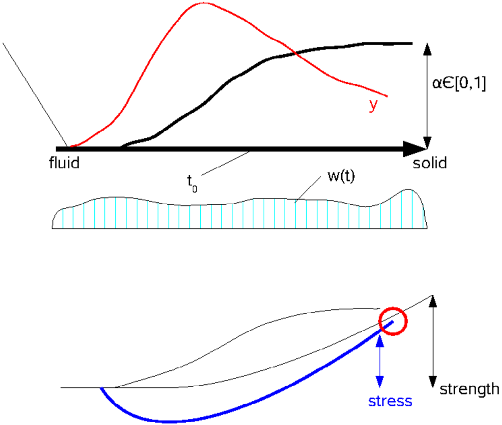
water cooling
$w(t) = q_7(t)$



State equation, parameter control

given: $q \in \mathbb{R}^k$
find: $u = (y, \tau)$ s.t.

$\Omega \subset \mathbb{R}^3$ polygonal
$y : \Omega \times [0, T] \to \mathbb{R}$ temperature
$\tau : \Omega \times [0, T] \to \mathbb{R}$ maturity

$$
\begin{cases}
\tau_t = g(y) & \text{in } \Omega \times (0, T) \\
\tau(0) = 0 & \text{in } \Omega \\
c(q)\rho(q)y_t - \lambda(q)\Delta y = Q_\infty(q)g(y)h(\tau, q) & \text{in } \Omega \times (0, T) \\
\lambda(q)\dfrac{\partial}{\partial n}y = \sigma(q)(\bar{y} - y) & \text{on } \Gamma \times (0, T) \\
y(0) = y_0(q) & \text{in } \Omega
\end{cases}
$$

short: $u = S(q)$

$g, h, \ldots$ model functions

State equation, time depedent control

given: $q \in L^2(0,T)$
find: $u = (y, \tau)$ s.t.

$\Omega \subset \mathbb{R}^3$ polygonal
$y : \Omega \times [0,T] \to \mathbb{R}$ temperature
$\tau : \Omega \times [0,T] \to \mathbb{R}$ maturity

$$\begin{cases} \tau_t = g(y) & \text{in } \Omega \times (0,T) \\ \tau(0) = 0 & \text{in } \Omega \\ c\rho y_t - \lambda \Delta y = Q_\infty g(y) h(\tau) - \dot{Q}_{pipe}(q(t)) & \text{in } \Omega \times (0,T) \\ \lambda \frac{\partial}{\partial n} y = \sigma(\bar{y} - y) & \text{on } \Gamma \times (0,T) \\ y(0) = y_0 & \text{in } \Omega \end{cases}$$

short: $u = S(q)$

$g, h, \ldots$ model functions

Simulation

- theory of single aspects well understood
- macroscopic approach with interpolated model functions for physical quantities $\Rightarrow$ model for simulation

- no optimization in literal sense yet
  $\Rightarrow$ formulation as parabolic optimal control problem with state constraints
- challenges: choice of discretisation, complex dynamic, nonlinear optimization problems with state constraints

$$\min J(q, u)$$
$$u = S(q)$$

- cost functional may contain
  - material / control costs $J_1(q)$
  - quality term $J_2(u)$
  - penalty costs for unwanted configurations

$$\min J(q,u)$$
$$u = S(q)$$
$$f(u) \geq 0$$

- cost functional may contain
  - material / control costs $J_1(q)$
  - quality term $J_2(u)$
  - penalty costs for unwanted configurations

- state constraint
  - temperature $u \leq u_b$
  - stability $f_{ct}(\tau) = f_{ct,\infty} \frac{\exp(a\tau^b) - \alpha_0}{1-\alpha_0} \geq f_{ct,given}$
  - cracking models

$$(P)\begin{cases} \min J(q,u) \\ A(q,u)(\phi) = 0 \\ f(u) \geq 0 \end{cases}$$

$q \in L^2(I,R)$ $u \in W(0,T)$

A.. weak formulation of parabolic pde

assume: optimality conditions hold

formulated via Lagrangian

$$\mathcal{L}(q,u,z,\mu) = J(q,u) - A(q,u)(z) + \langle f(u), \mu \rangle$$

$$(P)\begin{cases} \min J(q,u) \\ A(q,u)(\phi) = 0 \\ f(u) \geq 0 \end{cases}$$

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$$\mathcal{L}(q,u,z,\mu) = J(q,u) - A(q,u)(z) + \langle f(u),\mu\rangle$$

Regularization of the pointwise state constraint

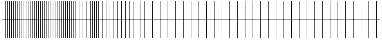
$\gamma \in \mathbb{R}_+, o > 1, b_\gamma(u) := \int\limits_Q \frac{1}{(o-1)\gamma^o}(f(u(t,x)))^{1-o}\mathrm{d}(t,x).$

$$(P_\gamma)\begin{cases} \min J_\gamma(q_\gamma,u_\gamma) := J(q_\gamma,u_\gamma) + b_\gamma(u_\gamma) \\ A(q_\gamma,u_\gamma)(\phi) = 0 \end{cases}$$

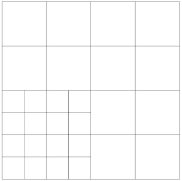
$\rightarrow$ opt.cond. via

$$\mathcal{L}_\gamma(q_\gamma,u_\gamma,z_\gamma) = J(q_\gamma,u_\gamma) - A(q_\gamma,u_\gamma)(z_\gamma) + b_\gamma(u_\gamma)$$

discretization of (P_γ)



implicit Euler in time, $I = \cup I_m$



linear elements in space

- full discretization
  $X_{kh}^{01} = \{v_\gamma^{kh} \in L^2(L^2) : v_{kh}|_{I_m} \in \mathcal{P}_0(I_m, V_h^1) \forall m = 1..M\}$
  with $V_h^1 = \{v_\gamma^h \in C(\bar{\Omega}) \cap H^1(\Omega) : v^h|_K \in \mathcal{Q}_1(K) \forall K \in \mathcal{T}_h\}$
- solve discrete problems $\rightarrow$ inexact Newton + LS-globalization on $j(q) = J(q, S(q))$
- balance refinement of spatial, temporal meshes + increase $\gamma \rightarrow \infty$
  $\rightarrow$ adaptively, driven by a posteriori error estimation

A-posteriori Error Estimation. Strategy

$$\boxed{J(q,u) - J(q_\gamma^{hk}, u_\gamma^{hk})}$$

$$\boxed{J(q,u) - J(q_\gamma^{hk}, u_\gamma^{hk})}$$

$$\swarrow \qquad \searrow$$

$$\boxed{J(q,u) - J(q_\gamma, u_\gamma)} + \boxed{J(q_\gamma, u_\gamma) - J(q_\gamma^{hk}, u_\gamma^{hk})}$$

$$J(q,u) - J(q_\gamma^{hk}, u_\gamma^{hk})$$

$$\swarrow \qquad \searrow$$

$$\boxed{J(q,u) - J(q_\gamma, u_\gamma)} + \boxed{J(q_\gamma, u_\gamma) - J(q_\gamma^{hk}, u_\gamma^{hk})}$$

$$\downarrow$$

$$J_\gamma(q_\gamma, u_\gamma) - J_\gamma(q_\gamma^{hk}, u_\gamma^{hk})$$
$$-b_\gamma(u_\gamma) + b_\gamma(u_\gamma^{hk})$$

$$J(q,u) - J(q_\gamma^{hk}, u_\gamma^{hk})$$

$$\swarrow \qquad \searrow$$

$$J(q,u) - J(q_\gamma, u_\gamma) \quad + \quad J(q_\gamma, u_\gamma) - J(q_\gamma^{hk}, u_\gamma^{hk})$$

$$\downarrow$$

$$J_\gamma(q_\gamma, u_\gamma) - J_\gamma(q_\gamma^{hk}, u_\gamma^{hk}) - b_\gamma(u_\gamma) + b_\gamma(u_\gamma^{hk})$$

$$\swarrow \qquad \searrow$$

$$J_\gamma(q_\gamma, u_\gamma) - J_\gamma(q_\gamma^{k}, u_\gamma^{k}) + b_\gamma(u_\gamma^{k}) - b_\gamma(u_\gamma) \quad + \quad J_\gamma(q_\gamma^{k}, u_\gamma^{k}) - J_\gamma(q_\gamma^{hk}, u_\gamma^{hk}) + b_\gamma(u_\gamma^{hk}) - b_\gamma(u_\gamma^{k})$$

$$\wr \qquad\qquad \wr \qquad\qquad \wr$$

$$\eta_\gamma \qquad\qquad \eta_k \qquad\qquad \eta_h$$

Let $\xi = (q, u, z, \mu)$ be (P)-optimal, $\hat{\xi}_\gamma = (q_\gamma, u_\gamma, z_\gamma)$ be (P_γ)-optimal, set $\xi_\gamma = (q_\gamma, u_\gamma, z_\gamma, b'_\gamma(u_\gamma))$.

error representation

$$
\begin{aligned}
J(q,u) - J(q_\gamma, u_\gamma) &= \mathcal{L}(\xi) - \mathcal{L}(\xi) + \langle f(u_\gamma), b'_\gamma(u_\gamma)\rangle \\
&= \frac{1}{2}\mathcal{L}'(\xi)(\xi - \xi_\gamma) + \frac{1}{2}\mathcal{L}'(\xi_\gamma)(\xi - \xi_\gamma) \\
&\quad + \langle f(u_\gamma), b'_\gamma(u_\gamma)\rangle + \mathcal{R} \\
&= \frac{1}{2}\langle f(u_\gamma) - f(u), b'_\gamma(u_\gamma)\rangle + \frac{1}{2}\langle f(u_\gamma), \mu\rangle + \mathcal{R}
\end{aligned}
$$

computable error estimator, heuristics

Let $\xi = (q, u, z, \mu)$ be (P)-optimal, $\hat{\xi}_\gamma = (q_\gamma, u_\gamma, z_\gamma)$ be (P_γ)-optimal, set $\xi_\gamma = (q_\gamma, u_\gamma, z_\gamma, b'_\gamma(u_\gamma))$.

error representation

$$\begin{aligned} J(q,u) - J(q_\gamma, u_\gamma) &= \mathcal{L}(\xi) - \mathcal{L}(\xi) + \langle f(u_\gamma), b'_\gamma(u_\gamma)\rangle \\ &= \frac{1}{2}\mathcal{L}'(\xi)(\xi - \xi_\gamma) + \frac{1}{2}\mathcal{L}'(\xi_\gamma)(\xi - \xi_\gamma) \\ &\quad + \langle f(u_\gamma), b'_\gamma(u_\gamma)\rangle + \mathcal{R} \\ &= \frac{1}{2}\langle f(u_\gamma) - f(u), b'_\gamma(u_\gamma)\rangle + \frac{1}{2}\langle f(u_\gamma), \mu\rangle + \mathcal{R} \end{aligned}$$

computable error estimator, heuristics

$\longrightarrow$ $\boxed{\eta_\gamma := c_0 \langle f(u_\gamma^{hk}), b'_\gamma(u_\gamma^{hk})\rangle}$

$c_0 \in [0.5, 1]$
depending on knowledge about $f(u)$

- techniques from error estimation for unconstrained problems

$$\eta_k := \frac{1}{2}\left(\tilde{\rho}^u(\cdot)(I_k^{(1)} z_\gamma^{hk} - z_\gamma^{hk}) + \tilde{\rho}^z(\cdot)(I_k^{(1)} u_\gamma^{hk} - u_\gamma^{hk})\right)$$
$$- b'_\gamma(\cdot)(I_k^{(1)} u_\gamma^{hk} - u_\gamma^{hk})$$

$$\eta_h := \frac{1}{2}\left(\tilde{\rho}^u(\cdot)(I_{2h}^{(2)} z_\gamma^{kh} - z_\gamma^{kh}) + \tilde{\rho}^z(\cdot)(I_{2h}^{(2)} u_\gamma^{kh} - u_\gamma^{kh})\right)$$
$$- b'_\gamma(\cdot)(I_{2h}^{(2)} u_\gamma^{kh} - u_\gamma^{kh})$$

Other contributions

- techniques from error estimation for unconstrained problems

$$\eta_k := \tfrac{1}{2}(\tilde{\rho}^u(\cdot)(I_k^{(1)} z_\gamma^{hk} - z_\gamma^{hk}) + \tilde{\rho}^z(\cdot)(I_k^{(1)} u_\gamma^{hk} - u_\gamma^{hk})) - b'_\gamma(\cdot)(I_k^{(1)} u_\gamma^{hk} - u_\gamma^{hk})$$

$$\eta_h := \tfrac{1}{2}(\tilde{\rho}^u(\cdot)(I_{2h}^{(2)} z_\gamma^{kh} - z_\gamma^{kh}) + \tilde{\rho}^z(\cdot)(I_{2h}^{(2)} u_\gamma^{kh} - u_\gamma^{kh})) - b'_\gamma(\cdot)(I_{2h}^{(2)} u_\gamma^{kh} - u_\gamma^{kh})$$

- error equilibration strategy
  relative contributions of $\eta_{abs} := |\eta_h| + |\eta_k| + |\eta_\gamma|$

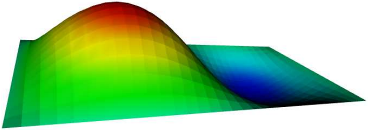
$\frac{\|\eta_h\|}{\eta_{abs}} > c_1$	$\frac{\|\eta_k\|}{\eta_{abs}} > c_1$	$\frac{\|\eta_\gamma\|}{\eta_{abs}} > c_1$
refine spatial mesh	refine temporal mesh	incrase γ, $\gamma_{new} := c_2\gamma$

e.g. $c_1 = 0.2, c_2 = 2$

$$\min \frac{1}{2}\|u - \sin(2\pi t)\sin(\pi x)\sin(\pi y)\|^2_{L^2(I\times\Omega)}$$

$$\begin{cases} u_t - \Delta u = 2\sin(\pi x)\sin(\pi y)q(t) & \text{in } I \times \Omega \\ u(t,x) = 0 & \text{on } I \times \Gamma \\ u(0,x) = 0 & \text{on } \Omega \end{cases}$$

$$u(t,x) \leq 0.1 \text{ in } I \times \Omega$$



$\Omega = [0,2] \times [0,1]$

$I = [0,1]$

$J(q^{hk},u^{hk}) - J(q^{hk}_\gamma,u^{hk}_\gamma)$	η_γ	$I_{eff,\gamma}$
-3.43e-03	-6.31e-03	0.543
-4.56e-04	-7.52e-04	0.607
-8.01e-05	-1.10e-04	0.728
-1.89e-05	-2.19e-05	0.863
-5.33e-06	-5.63e-06	0.947
-1.62e-06	-1.65e-06	0.982
-5.06e-07	-5.09e-07	0.994
-1.60e-07	-1.60e-07	0.998
-5.04e-08	-5.05e-08	0.999
-1.60e-08	-1.60e-08	1.000
-5.05e-09	-5.05e-09	1.000

$$\min \frac{1}{2}\|u - \sin(2\pi t)\sin(\pi x)\sin(\pi y)\|^2_{L^2(I\times\Omega)}$$

$$\begin{cases} u_t - \Delta u = 2\sin(\pi x)\sin(\pi y)q(t) & \text{in } I\times\Omega \\ u(t,x) = 0 & \text{on } I\times\Gamma \\ u(0,x) = 0 & \text{on } \Omega \end{cases}$$

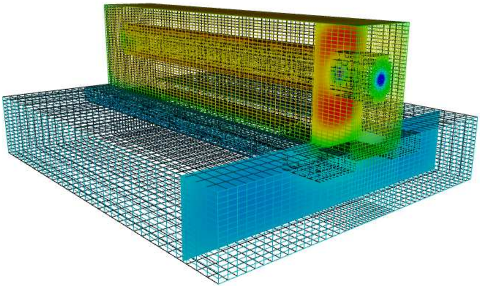
$$u(t,x) \leq 0.1 \text{ in } I\times\Omega$$

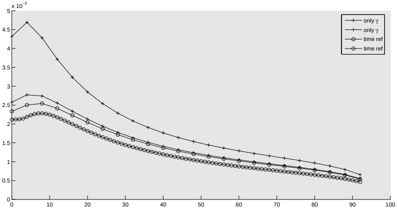
N	M	γ	η_h	η_k	η_γ	$J(q,u)^* - J_\gamma(q_\gamma^{hk}, u_\gamma^{hk})$	I_{eff}
45	6	3.16	2.50e-02	2.55e-06	-4.65e+00	-2.34e+00	0.507
45	6	10	2.56e-02	-7.12e-04	-4.71e-01	-2.52e-01	0.538
45	6	31.6	-7.53e-05	-2.19e-03	-5.49e-02	-3.30e-02	0.578
45	6	100	-1.31e-03	-2.43e-03	-6.35e-03	-6.91e-03	0.685
45	12	316	-1.56e-03	-9.21e-04	-8.08e-04	-3.36e-03	1.022
153	24	316	-3.90e-04	-3.63e-04	-8.83e-04	-1.39e-03	0.848
561	48	1000	-1.06e-04	-1.38e-04	-1.17e-04	-3.43e-04	0.949
2145	96	3160	-7.96e-05	-5.33e-05	-1.34e-05	-1.46e-04	1.000

water cooling

find $q \in Q = L^2(0, T)$

s.t. $u \leq u_b$





$I_{eff,\gamma} =$
$0.731, 0.761, 1.011, \ldots$

Summary

optimal control problem: parabolic + state constraints

- solution via regularization of constraints
- a-posteriori estimation of error contributions
- equilibration strategy for η_h, η_k, η_γ

mechanical properties of young concrete

- formulation as OCP in above problem class
- first optimization results
- cooperation with:

  T. Apel, T. Flaig (UniBW Munich)
  R. Nothnagel (ibmb Braunschweig)

Thank you

for your attention!