Convergence Analysis of Monte Carlo Calibration of Financial Market Models

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Workshop on PDE Constrained Optimization of Certain and Uncertain Processes

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Fundamentals

Focus will be on calibration of European call options

Definition 1 (European Call Option)

A **European call option** is the right to buy a predetermined underlying (e.g. stock) at a certain time T (maturity) for a certain price K (strike).

Fundamentals

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Definition 1 (European Call Option)

A **European call option** is the right to buy a predetermined underlying (e.g. stock) at a certain time T (maturity) for a certain price K (strike).

Definition 2 (Price of a Call Option)

The **price of a call option** C in $t = 0$ can be calculated through

$$C = e^{-rT} E\left(\max(S_T - K, 0)\right)$$

where r is the risk free rate and S_T the value of the underlying at future time T.

Stochastic Differential Equation

L-dimensional system of stochastic differential equations (SDE):

$$dY_t(x) = a(x, Y_t(x))dt + b(x, Y_t(x))dW_t$$

where

$x \in \mathbb{R}^P$	vector of parameters
$Y_t = [S_t, Y_t^2, ..., Y_t^L] \in \mathbb{R}^L$	Solution of SDE
$W_t = (W_t^1, ..., W_t^L) \in \mathbb{R}^L$	Vector of Brownian motions
$a : \mathbb{R}^P \times \mathbb{R}^L \to \mathbb{R}^L$	$a^l(x, Y_t(x))dt$
$b : \mathbb{R}^P \times \mathbb{R}^L \to \mathbb{R}^L \times \mathbb{R}^L$	$\sum_{\nu=1}^{L} b^{l,\nu}(x, Y_t(x))dW_t^\nu$, $l = 1, ..., L$

Least Squares Problem

Continuous Optimization Problem (True Problem)

$$\min_{x \in X} f(x) := \sum_{i=1}^{I} \left(C^i(x) - C^i_{\text{obs}} \right)^2$$

where $C^i(x) = e^{-rT_i} E \left(\max(S_{T_i}(x) - K_i, 0) \right)$

s.t. $dY_t(x) = a(x, Y_t(x))dt + b(x, Y_t(x))dW_t, \; Y_0 > 0$

$X \subset \mathbb{R}^P$ convex and compact

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$X \subset \mathbb{R}^P$ convex and compact

Discretized Optimization Problem (SAA Problem)

$$\min_{x \in X} f_{M,\Delta t,\epsilon} := \sum_{i=1}^{I} \left(C^i_{M,\Delta t,\epsilon}(x) - C^i_{\text{obs}} \right)^2$$

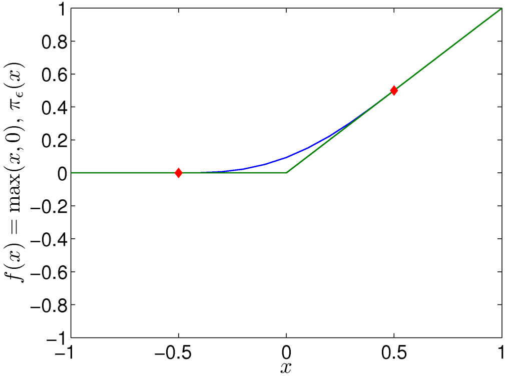
where $C^i_{M,\Delta t,\epsilon}(x) := e^{-rT_i} \frac{1}{M} \sum_{m=1}^{M} \left(\pi_\epsilon(s^m_{N_i,\epsilon}(x) - K_i) \right)$

s.t. $y^m_{n+1,\epsilon}(x) = y^m_{n,\epsilon}(x) + a_\epsilon(x, y^m_{n,\epsilon}(x))\Delta t_n + b_\epsilon(x, y^m_{n,\epsilon}(x))\Delta W^m_n$

Smoothing Non-differentiabilities

Consider Heston's Model:

$$
\begin{aligned}
dS_{t,\epsilon} &= (r-\delta)S_{t,\epsilon}dt + \sqrt{v_{t,\epsilon}^{+}}S_{t,\epsilon}dW_t^1 \\
dv_{t,\epsilon} &= \kappa(\theta - v_{t,\epsilon}^{+})dt + \sigma\sqrt{v_{t,\epsilon}^{+}}(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2)
\end{aligned}
$$



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Convergence

True Problem

$$\min_{x \in X} f(x) := \sum_{i=1}^{I} \left(C^i(x) - C^i_{\text{obs}}\right)^2$$

SAA Problem

$$\min_{x \in X} f_{M_k,\Delta t_k,\epsilon_k} := \sum_{i=1}^{I} \left(C^i_{M_k,\Delta t_k,\epsilon_k}(x) - C^i_{\text{obs}}\right)^2$$

$$\left.\begin{array}{ll} \text{Increase number of simulations:} & M_k \uparrow \infty \\ \text{Decrease discretization step size:} & \Delta t_k \downarrow 0 \\ \text{Decrease smoothing parameter:} & \epsilon_k \downarrow 0 \end{array}\right\} x_k \in X \text{ solutions}$$

Convergence

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X compact $\Rightarrow x_{k_l} \to x^*$ with x^* in X.

Convergence

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X compact $\Rightarrow x_{k_l} \to x^*$ with x^* in X.

Question

x^* solution of the true problem?

Local Minima

$$\min_{x\in[-1;1]} f(x) := x^2$$

$$\min_{x\in[-1;1]} f_M(x) := x^2 - M^{-1}\sin(Mx^2)$$

Local Minima

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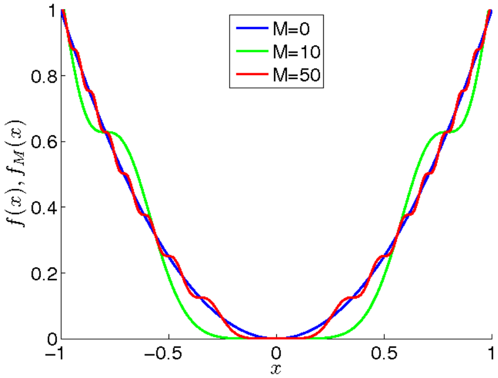
Local minima might lead to problems:



Literature Review

True Problem:

$$\min_{x\in X} h(x) := E(H(x,\omega))$$

SAA Problem

$$\min_{x\in X} h_{\mathsf{M}}(x) := \frac{1}{M}\sum_{m=1}^{M} H(x,\omega_m)$$

- Shapiro (2000): Convergence if $\min h(x)$ produces global minimum
- Rubinstein & Shapiro (1993): Convergence to a critical first order point under assumption that $H(x,\omega)$ is dominated integrable and continuous
- Bastin *et al.* (2006): Additionally second order convergence even for stochastic constraints

$\Rightarrow$ Dependence on three error sources: Monte Carlo, discretization and smoothing!

Goal: First Order Optimality

Steps to be taken:

1. Pathwise Uniqueness of SDE
2. Uniform Convergence:

$$\lim_{k\to\infty} \sup_{x\in X} |f_{\mathsf{M}_k,\Delta t_k,\epsilon_k}(x) - f(x)| = 0$$

$$\lim_{k\to\infty} \sup_{x\in X} \|\nabla f_{\mathsf{M}_k,\Delta t_k,\epsilon_k}(x) - \nabla f(x)\| = 0$$

3. First Order Optimality Condition:

$$\nabla f(x^*)^T (x - x^*) \geq 0 \quad \forall x \in X$$

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Pathwise Uniqueness under Lipschitz Continuity

Theorem 3 (Kloeden & Platen)

Under the assumptions that

There exists a constant $K_{\text{Lip}} > 0$ such that $\forall t \in [0,T]$ and $y \in \mathbb{R}^L$
$|a(t,y) - a(t,z)| + |b(t,y) - b(t,z)| \leq K_{\text{Lip}}|y - z|$

There exists a constant $K_{\text{Grow}} > 0$ such that $\forall t \in [0,T]$ and $y \in \mathbb{R}^L$
$|a(t,y)| + |b(t,y)| \leq K_{\text{Grow}}(1 + |y|)$

the stochastic differential equation

$$dY_t = a(t,Y_t)dt + b(t,Y_t)dW_t\,,\ Y_0 \in (0,\infty).$$

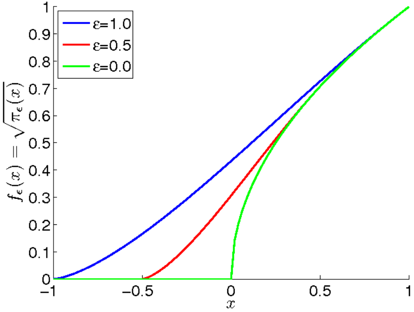
has a pathwise unique strong solution Y_t on $[0,T]$.

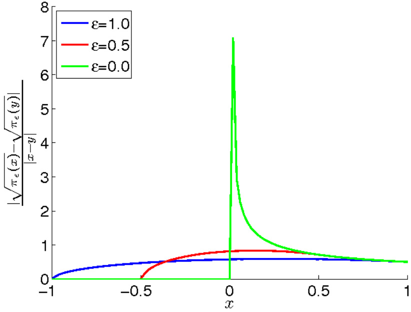
Problem: Lipschitz Continuity

Consider Heston's model

$$
\begin{aligned}
dS_{t,\epsilon} &= (r-\delta)S_{t,\epsilon}dt + \sqrt{\pi_\epsilon(v_{t,\epsilon})}S_{t,\epsilon}dW_t^1 \\
dv_{t,\epsilon} &= \kappa(\theta - \pi_\epsilon(v_{t,\epsilon}))dt + \sigma\sqrt{\pi_\epsilon(v_{t,\epsilon})}(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2)
\end{aligned}
$$

Lipschitz continuity for $\epsilon > 0$





Yamada Condition

Theorem 4

Let

$$dY_{t,\epsilon} = a_\epsilon(t, Y_{t,\epsilon})dt + b_\epsilon(t, Y_{t,\epsilon})dW_t.$$

with

$$\begin{aligned} a_\epsilon(t, Y_{t,\epsilon}) &= (a_\epsilon^1(t, Y_{t,\epsilon}^1), ..., a_\epsilon^L(t, Y_{t,\epsilon}^L))^T \\ b_\epsilon(t, Y_{t,\epsilon}) &= \mathsf{diag}(b_\epsilon^1(t, Y_{t,\epsilon}^1), ..., b_\epsilon^L(t, Y_{t,\epsilon}^L)) \end{aligned}$$

If there exists a positive increasing function $\beta : [0, \infty) \to [0, \infty)$ *with*

$$|b^i(t, x) - b^i(t, y)| \leq \beta(|x - y|) \quad \forall x, y \in \mathbb{R}, \quad i = 1, ..., L$$

and

$$\int_0^\delta \beta^{-2}(z)dz = \infty.$$

with an arbitrarily small $\delta > 0$ *...*

Yamada Condition (2)

... and a positive increasing concave function $\alpha : [0,\infty) \to [0,\infty)$ such that

$$|a^i(t,x) - a^i(t,y)| \leq \alpha(|x-y|) \quad \forall x,y \in \mathbb{R}, \quad i = 1,...,L$$

with

$$\int_0^\delta \alpha^{-1}(z)dz = \infty.$$

with an arbitrarily small $\delta > 0$, the SDE has a pathwise unique solution.

Proof: Yamada (1971)

Yamada Condition (3)

Reconsider Heston's model

$$\begin{aligned} dS_{t,\epsilon} &= (r-\delta)S_{t,\epsilon}dt + \sqrt{\pi_\epsilon(v_{t,\epsilon})}S_{t,\epsilon}dW_t^1 \\ dv_{t,\epsilon} &= \kappa(\theta - \pi_\epsilon(v_{t,\epsilon}))dt + \sigma\sqrt{\pi_\epsilon(v_{t,\epsilon})}(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2) \end{aligned}$$

The drift is Lipschitz continuous:

$$|a^i(t,x) - a^i(t,y)| \le K_{\mathsf{Lip}}|x-y| \quad \forall x,y \in \mathbb{R}, \quad i=1,2$$

and the diffusion is Hölder continuous:

$$|b^i(t,x) - b^i(t,y)| \le \sqrt{|x-y|} \quad \forall x,y \in \mathbb{R}, \quad i=1,2$$

with

$$\int_0^\delta \frac{1}{K_{\mathsf{Lip}}z}dz = \infty; \quad \int_0^\delta \frac{1}{z}dz = \infty.$$

Problem: Independent components required

Heston's model:

$$\begin{aligned} dS_{t,\epsilon} &= (r-\delta)S_{t,\epsilon}dt + \sqrt{\pi_\epsilon(v_{t,\epsilon})}S_{t,\epsilon}dW_t^1 \\ dv_{t,\epsilon} &= \kappa(\theta - \pi_\epsilon(v_{t,\epsilon}))dt + \sigma\sqrt{\pi_\epsilon(v_{t,\epsilon})}dW_t^2 \end{aligned}$$

Problem: Independent components required

Heston's model:

$$\begin{aligned} dS_{t,\epsilon} &= (r-\delta)S_{t,\epsilon}dt + \sqrt{\pi_\epsilon(v_{t,\epsilon})}S_{t,\epsilon}dW_t^1 \\ dv_{t,\epsilon} &= \kappa(\theta - \pi_\epsilon(v_{t,\epsilon}))dt + \sigma\sqrt{\pi_\epsilon(v_{t,\epsilon})}dW_t^2 \end{aligned}$$

Solution:

- Process $v_{t,\epsilon}$ has pathwise unique solution following Yamada's Theorem
- Insert this unique solution in process $S_{t,\epsilon}$
- Process $S_{t,\epsilon}$ has pathwise unique solution following Yamada's Theorem

$\Rightarrow$ Pathwise unique solution via Yamada's Theorem

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Convergence of the Problem

Reconsider:

$$
\begin{aligned}
|f_{\mathsf{M},\Delta t,\epsilon}(x) - f(x)| &\leq |f_{\mathsf{M},\Delta t,\epsilon}(x) - f_{\Delta t,\epsilon}(x)| && (1)\\
&+ |f_{\Delta t,\epsilon}(x) - f_{\epsilon}(x)| && (2)\\
&+ |f_{\epsilon}(x) - f(x)| && (3)
\end{aligned}
$$

Convergence of the Problem

Reconsider:

$$\begin{aligned} |f_{\mathsf{M},\Delta t,\epsilon}(x) - f(x)| &\leq |f_{\mathsf{M},\Delta t,\epsilon}(x) - f_{\Delta t,\epsilon}(x)| && (1)\\ &+ |f_{\Delta t,\epsilon}(x) - f_{\epsilon}(x)| && (2)\\ &+ |f_{\epsilon}(x) - f(x)| && (3) \end{aligned}$$

Assumption:

There exists a constant $K_{\mathsf{Grow}} > 0$ such that $\forall t \in [0,T]$ and $y \in \mathbb{R}^L$ $\|a_\epsilon(t,y)\| + \|b_\epsilon(t,y)\| \leq K_{\mathsf{Grow}}(1 + \|y\|)$.

Convergence of Smoothed and Discretized SDE

Theorem 5

Consider the SDE

$$dY_{t,\epsilon} = a_\epsilon(t, Y_{t,\epsilon})dt + b_\epsilon(t, Y_{t,\epsilon})dW_t.$$

and the continuously interpolated process

$$y_{t,\epsilon} = Y_0 + \int_0^t a_\epsilon(x, y_{\tau(s),\epsilon})ds + \int_0^t b_\epsilon(x, y_{\tau(s),\epsilon})dW_s$$

where $\tau(s) = n, \forall s \in [\tau_n, \tau_{n+1})$ *and* $n = 0, ..., N-1.$ *Assuming that the growth condition holds and the SDE has a pathwise unique solution it holds*

$$\lim_{\Delta t \to 0} \sup_{x \in X} E\left(|y_{t,\epsilon} - Y_{T,\epsilon}|^2\right) = 0.$$

Proof: Kaneko & Nakao (1988)

Convergence of Smoothed SDE

Theorem 6

Assume that the growth condition and the pathwise uniqueness holds for a solution of

$$dY_t = a(t, Y_t)dt + b(t, Y_t)dW_t$$

and let $Y_{t,\epsilon}$ be a solution of

$$dY_{t,\epsilon} = a_\epsilon(t, Y_{t,\epsilon})dt + b_\epsilon(t, Y_{t,\epsilon})dW_t.$$

If a_ϵ and b_ϵ converge uniformly to a and b for $\epsilon \to 0$, i.e.

$$\lim_{\epsilon \to 0} \sup_{t \in [0,T]} \sup_{x \in X} \left(|a_\epsilon(t,x) - a(t,x)| + \|b_\epsilon(t,x) - b(t,x)\| \right) = 0.$$

where $\| \cdot \|$ is a matrix norm, it holds

$$\lim_{\epsilon \to 0} \sup_{x \in X} E\left(|Y_{t,\epsilon} - Y_t|^2 \right) = 0.$$

Proof: Kaneko & Nakao (1988)

Dominated Integrability & Continuity

Lemma 7

Assume that the families $\{\pi(S_T(x,\omega) - K), x \in X\}$ *are dominated by a Q-integrable function* $\overline{P}(\omega)$*. Then there exist* $\overline{\Delta t} > 0$ *and* $\overline{\epsilon} > 0$ *such that* $\{\pi_\epsilon(s_{N,\epsilon}(x,\omega) - K), x \in X\}$ *is dominated by a Q-integrable function for all* $\Delta t \in [0, \overline{\Delta t}]$ *and* $\epsilon \in [0, \overline{\epsilon}]$*.*

Lemma 8

If the functions $\pi(S_T(\cdot,\omega) - K)$ *are continuous on* X *for Q almost every* ω*, the functions* $\pi_\epsilon(s_{N,\epsilon}(x,\omega) - K)$ *are continuous on* X *for* $0 < \Delta t < \infty$ *and* $0 < \epsilon < \infty$*.*

Uniform Convergence

Theorem 9

Assume that the families $\{\pi(S_T(x,\omega)-K), x\in X\}$ *are dominated by a Q-integrable function* $\overline{P}(\omega)$ *and furthermore the functions* $\pi(S_T(\cdot,\omega)-K)$ *are continuous on X for* Q *almost every* ω*. If additionally X is compact, then* $f(x)$ *is continuous on X. Furthermore* $f_{\mathsf{M},\Delta t,\epsilon}$ *converges uniformly to* f *on X, i.e. for given sequences* $(M_k)_k \subset \mathbb{N}$*,* $(\Delta t_k)_k \subset \mathbb{R}_+$ *and* $(\epsilon_k)_k \subset \mathbb{R}_+$ *satisfying* $M_k \uparrow \infty$*,* $\Delta t_k \downarrow 0$*,* $\epsilon_k \downarrow 0$ *it holds*

$$\lim_{k\to\infty} \sup_{x\in X} |f_{\mathsf{M}_k,\Delta t_k,\epsilon_k}(x) - f(x)| = 0.$$

Note that the same can be shown for the gradients!

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First Order Optimality Condition

Theorem 10

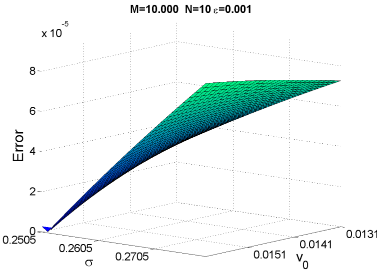
Assume that the families $\{\pi(S_T(x,\omega) - K), x \in X\}$ *and* $\{\frac{\partial}{\partial x_p}\pi(S_T(\cdot,\omega) - K), x \in X\}$, $i = 1, ..., I$ *are dominated by a Q-integrable function* $\overline{P}(\omega)$ *and furthermore the functions* $\pi(S_T(\cdot,\omega) - K)$ *and* $\frac{\partial}{\partial x_p}\pi(S_T(\cdot,\omega) - K)$, $i = 1, ..., I$ *are continuous on X for* Q *almost every* ω *and additionally that X is compact. Further let* $(M_k)_k \subset \mathbb{N}_+$, $(\Delta t_k)_k \subset \mathbb{R}_+$, $(\epsilon_k)_k \subset \mathbb{R}_+$ *and* $(\gamma_k)_k \subset \mathbb{R}_+$ *with* $\mathsf{M}_k \uparrow \infty$, $\Delta t_k \downarrow 0$, $\epsilon_k \downarrow 0$ *and* $\gamma_k \downarrow 0$ *be given sequences and assume that* $(x_k)_{k \in \mathbb{N}} \subset X$ *is a sequence of points satisfying*

$$\nabla f(x_k)^T (x - x_k) \geq -\gamma_k \quad \forall\, x \in X.$$

Then every limit point $x^* \in X$ *of* $(x_k)_k$ *almost surely satisfies the first order optimality condition*

$$\nabla f(x^*)^T (x - x^*) \geq 0 \quad \forall\, x \in X.$$

Convergence: Graphical Illustration



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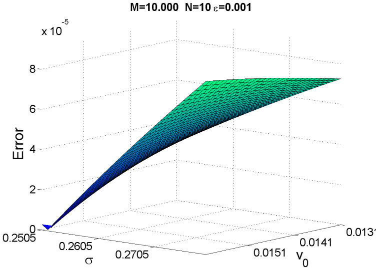

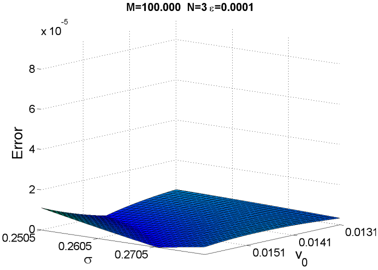

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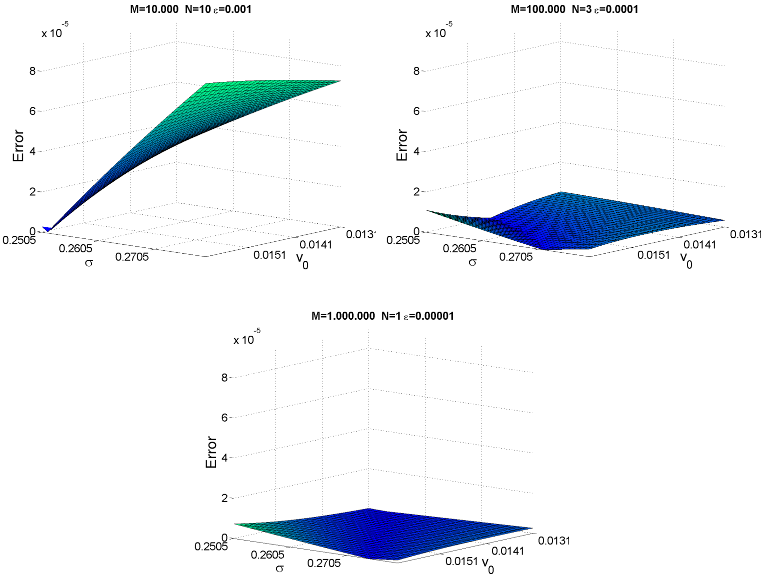

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Conclusions

- Set up calibration problem
- Discretized via Monte Carlo, Euler-Maruyama and smoothing
- Pathwise Uniqueness for resulting SDE under Yamada Condition
- Uniform convergence of objectives under unrestrictive assumptions
- First order optimality condition satisfied for limit point x^*

Bibliography

- Bastin,F., Cirillo,C. and Toint,P.L.:
  Convergence Theory for Nonconvex Stochastic Programming with an Application to Mixed Logit, Mathematical Programming Series B, Vol. 108, 2006,

- Rubinstein,R.Y. and Shapiro,A.:
  Discrete Event Systems, John Wiley, 1993

- Shapiro, A.:
  Stochastic Programming by Monte Carlo Simulation Methods, Stochastic Programming E-Print Series 2000,

- Kaneko,H. and Nakao,S.:
  A Note on Approximation for Stochastic Differential Equations, Seminaire de Probabilites, XXII, Lecture Notes in Mathematics, Vol. 1321, 1988,

- Yamada, T. and Watanabe, S.:
  On the Uniqueness of Solutions of Stochastic Differential Equations, Journal of Mathematics of Kyoto University, Vol 11, 1971

- Kaebe, C., Maruhn, J. and Sachs, E.W.:
  Adjoint Based Monte Carlo Calibration of Financial Market Models, Journal of Finance and Stochastics (to appear)