A Boundary Element Energy Approximation of Dirichlet Control Problems

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Dirichlet Boundary Control Problem

Boundary Integral Equations

Boundary Element Solution

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Dirichlet Boundary Control Problem

Model problem: Find $(u, z) \in H^1(\Omega) \times \mathcal{A}$ minimizing

$$J(u, z) \;=\; \frac{1}{2} \int\limits_{\Omega} [u(x) - \overline{u}(x)]^2 dx + \frac{1}{2} \varrho \|z\|_A^2$$

subject to the constraint

$$-\Delta u(x) \;=\; f(x) \quad \text{for } x \in \Omega, \qquad u(x) \;=\; z(x) \quad \text{for } x \in \Gamma = \partial\Omega,$$

where

- given target $\overline{u} \in L_2(\Omega)$
- given volume density $f \in L_2(\Omega)$
- fixed parameter $\varrho \in \mathbb{R}_+$
- bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ with boundary $\Gamma = \partial\Omega$
- A is a suitable operator to be chosen where $\| \cdot \|_A^2 = \langle A \cdot, \cdot \rangle_\Gamma$

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- $\mathcal{A} = L_2(\Omega)$: [Casas, Raymond 06], [May, Rannacher, Vexler 08], [Deckelnick, Günther, Hinze 08]

Solving the State Equation

Let $u_f \in H_0^1(\Omega)$ be the weak particular solution of

$$-\Delta u_f(x) = f(x) \quad \text{for } x \in \Omega, \quad u_f(x) = 0 \quad \text{for } x \in \Gamma.$$

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Let $u_z = \mathcal{S}z \in H^1(\Omega)$ be the weak solution of

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$$u_z \in H^1(\Omega) \Rightarrow u_{z|\Gamma} \in H^{1/2}(\Gamma) \Rightarrow \mathcal{S} : H^{1/2}(\Gamma) \to H^1(\Omega) \subset L_2(\Omega).$$

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is given by

$$u = \mathcal{S}z + u_f.$$

Reduced Cost Functional

Using $u = \mathcal{S}z + u_f$: Find the minimizer $z \in H^{1/2}(\Gamma)$ of the reduced cost functional

$$\widetilde{J}(z) \;=\; \frac{1}{2}\int\limits_{\Omega}[(\mathcal{S}z)(x) + u_f(x) - \overline{u}(x)]^2 dx + \frac{1}{2}\varrho\,\langle Az, z\rangle_\Gamma.$$

$\widetilde{J}(\cdot)$ is convex $\Rightarrow$ minimizer z determined by optimality condition

$$\mathcal{S}^*\mathcal{S}z + \mathcal{S}^*(u_f - \overline{u}) + \varrho Az \;=\; 0.$$

$\mathcal{S}^* : L_2(\Omega) \to H^{-1/2}(\Gamma)$ adjoint operator of $\mathcal{S} : H^{1/2}(\Gamma) \to L_2(\Omega)$, i.e.,

$$\langle \mathcal{S}^*\psi, \varphi\rangle_\Gamma = \langle \psi, \mathcal{S}\varphi\rangle_\Omega = \int\limits_{\Omega}\psi(x)(\mathcal{S}\varphi)(x)dx \quad \text{for all } \varphi \in H^{1/2}(\Gamma), \psi \in L_2(\Omega).$$

Optimality system

Optimality condition: $\mathcal{S}^*\mathcal{S}z + \mathcal{S}^*(u_f - \overline{u}) + \varrho A z = 0.$

- primal variable $u = \mathcal{S}z + u_f$
- adjoint variable $\tau = \mathcal{S}^*(u - \overline{u}) \in H^{-1/2}(\Gamma)$,

$\Rightarrow$ coupled problem

$$\tau + \varrho A z = 0, \quad \tau = \mathcal{S}^*(u - \overline{u}), \quad u = \mathcal{S}z + u_f$$

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Adjoint variable $\tau = \mathcal{S}^*(u - \overline{u})$ is Neumann datum

$$\tau(x) = -\frac{\partial}{\partial n_x} p(x) \quad \text{for } x \in \Gamma$$

of the adjoint Dirichlet boundary value problem

$$-\Delta p(x) = u(x) - \bar{u}(x) \quad \text{for } x \in \Omega, \quad p(x) = 0 \quad \text{for } x \in \Gamma.$$

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$\Rightarrow$ optimality condition: $\dfrac{\partial}{\partial n_x} p(x) = \varrho(Az)(x) \quad \text{for } x \in \Gamma.$

Choosing $\mathcal{A}$ and $\|\cdot\|_A$

Optimality condition: $$\frac{\partial}{\partial n_x} p(x) \;=\; \varrho(Az)(x) \quad \text{for } x \in \Gamma.$$

$$u \in H^1(\Omega) \Rightarrow u_{|\Gamma} = z \in H^{1/2}(\Gamma)$$

$\rightarrow$ elliptic, self–adjoint, and bounded operator $A : H^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)$,

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$\to$ elliptic, self–adjoint, and bounded operator $A : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, e.g., the stabilized hypersingular boundary integral operator $A = \widetilde{D}$

$$\langle \widetilde{D}z, w\rangle_\Gamma := \langle Dz, w\rangle_\Gamma + \langle z, 1\rangle_\Gamma \langle w, 1\rangle_\Gamma \quad \text{for all } z, w \in H^{1/2}(\Gamma)$$

where

$$(Dz)(x) = -\frac{\partial}{\partial n_x} \int_\Gamma \frac{\partial}{\partial n_y} U^*(x, y) z(y) ds_y \quad \text{for } x \in \Gamma,$$

and the fundamental solution

$$U^*(x, y) = \begin{cases} -\dfrac{1}{2\pi} \log |x - y| & \text{for } n = 2, \\ \dfrac{1}{4\pi} \dfrac{1}{|x - y|} & \text{for } n = 3 \end{cases}$$

Unique Solvability and Regularity

Theorem

Let $A : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ *be elliptic, self–adjoint, and bounded. The operator* $T_\varrho = \varrho A + \mathcal{S}^*\mathcal{S} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ *is bounded and* $H^{1/2}(\Gamma)$*–elliptic.* $\Rightarrow$ *unique solvability of*

$$T_\varrho z := (\mathcal{S}^*\mathcal{S} + \varrho A)z = \mathcal{S}^*(\bar{u} - u_f) =: g\,.$$

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Optimality system

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \quad u = z \quad \text{on } \Gamma, \\ -\Delta p &= u - \overline{u} && \text{in } \Omega, \quad p = 0 \quad \text{on } \Gamma, \\ \frac{\partial}{\partial n_x} p &= \varrho A z && \text{on } \Gamma. \end{aligned}$$

Regularity:

If Γ smooth and $u - \overline{u} \in H^s(\Omega)$: $\Rightarrow p \in H^{s+2}(\Omega) \Rightarrow \frac{\partial}{\partial n} p \in H^{s+1/2}(\Gamma)$

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By $A : H^{s+1/2}(\Gamma) \to H^{s+3/2}(\Gamma)$: $z \in H^{s+3/2}(\Gamma) \Rightarrow u \in H^{s+2}(\Omega)$.

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If Γ piecewise polygonal: reduced regularity.

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Primal Boundary Value Problem

$$-\Delta u(x) = f(x) \quad \text{for } x \in \Omega, \quad u(x) = z(x) \quad \text{for } x \in \Gamma$$

Its solution is given by the representation formula for $\widetilde{x} \in \Omega$,

$$u(\widetilde{x}) = \int_\Gamma U^*(\widetilde{x}, y) \frac{\partial}{\partial n_y} u(y) ds_y - \int_\Gamma \frac{\partial}{\partial n_y} U^*(\widetilde{x}, y) z(y) ds_y + \int_\Omega U^*(\widetilde{x}, y) f(y) dy,$$

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For given $z \in H^{1/2}(\Gamma)$, we determine $t = \frac{\partial}{\partial n} u \in H^{-1/2}(\Gamma)$ by

$$(Vt)(x) = (\frac{1}{2} I + K) z(x) - (N_0 f)(x) \quad \text{for } x \in \Gamma.$$

where

$$(Vt)(x) = \int_\Gamma U^*(x, y) t(y) ds_y \qquad (Kz)(x) = \int_\Gamma \frac{\partial}{\partial n_y} U^*(x, y) z(y) ds_y$$

$$(N_0 f)(x) = \int_\Omega U^*(x, y) f(y) dy \quad \text{for } x \in \Gamma$$

Adjoint Boundary Value Problem

$$-\Delta p(x) = u(x) - \bar{u}(x) \quad \text{for } x \in \Omega, \quad p(x) = 0 \quad \text{for } x \in \Gamma,$$

We obtain the unknown Neumann datum $q = \frac{\partial}{\partial n} p$ by

$$(Vq)(x) = (N_0 \bar{u})(x) - (N_0 u)(x) \quad \text{for } x \in \Gamma$$

BUT: Unknown u occurs in domain potential.

Adjoint Boundary Value Problem

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BUT: Unknown u occurs in domain potential.
Remedy: integration by parts $(-\Delta u = f)$ to end up with a system of boundary integral equations only:

$$\int_\Omega U^*(\widetilde{x}, y) u(y) dy = \int_\Omega [\Delta_y V^*(\widetilde{x}, y)] u(y) dy$$

$$= \int_\Gamma \frac{\partial}{\partial n_y} V^*(\widetilde{x}, y) u(y) ds_y - \int_\Gamma V^*(\widetilde{x}, y) \frac{\partial}{\partial n_y} u(y) ds_y + \int_\Omega V^*(\widetilde{x}, y) [\Delta u(y)] dy$$

where $V^*(x, y)$ is a is the fundamental solution of the Bi–Laplacian.

Adjoint Boundary Value Problem

Therefore we get

$$(Vq)(x) = (V_1 t)(x) - (K_1 z)(x) + (N_0 \bar{u})(x) + (M_0 f)(x) \quad \text{for } x \in \Gamma.$$

where

$$(V_1 t)(x) = \int_\Gamma V^*(x,y)t(y)ds_y \quad (K_1 z)(x) = \int_\Gamma \frac{\partial}{\partial n_y} V^*(x,y)z(y)ds_y$$

$$(M_0 f)(x) = \int_\Omega V^*(x,y)f(y)dy \quad \text{for } x \in \Gamma$$

are integral operators of the bi-harmonic equation.

Optimality Condition

To obtain a symmetric system, we rewrite $q(x) = \frac{\partial}{\partial n} p(x)$ by integral operators

$$q(x) = (\frac{1}{2} I + K')q(x) - (D_1 z)(x) - (K_1' t)(x) - (N_1 \bar{u})(x) - (M_1 f)(x)$$

where $(x \in \Gamma)$

$$(N_1 \bar{u})(x) = \lim_{\Omega \ni \widetilde{x} \to x \in \Gamma} n_x \cdot \nabla_{\widetilde{x}} \int_\Omega U^*(\widetilde{x}, y) \bar{u}(y) dy$$

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Inserted into the optimality condition:

$$\varrho(Az)(x) = (\frac{1}{2} I + K')q(x) - (D_1 z)(x) - (K_1' t)(x) - (N_1 \bar{u})(x) - (M_1 f)(x).$$

Coupled System

Find $(z, t, q) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma)$:

$$\begin{pmatrix} -V_1 & V & K_1 \\ V & & -\frac{1}{2}I - K \\ K_1' & -\frac{1}{2}I - K' & \varrho A + D_1 \end{pmatrix} \begin{pmatrix} t \\ q \\ z \end{pmatrix} = \begin{pmatrix} N_0 \bar{u} + M_0 f \\ -N_0 f \\ -N_1 \bar{u} - M_1 f \end{pmatrix}.$$

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The Schur complement system is uniquely solvable

$$T_\varrho z = g.$$

Theorem

Let $A : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ be elliptic, self–adjoint, and bounded. The composed boundary integral operator

$$T_\varrho = \varrho A + D_1 - (\frac{1}{2}I + K')V^{-1}V_1V^{-1}(\frac{1}{2}I + K) + K_1'V^{-1}(\frac{1}{2}I + K) + (\frac{1}{2}I + K')V^{-1}K_1$$

is self–adjoint, bounded, and $H^{1/2}(\Gamma)$–elliptic.

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Galerkin Boundary Element Formulation

Approximate

- t by piecewise constant basis functions ψ_k
- q by piecewise constant basis functions ψ_k
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System of linear equations:

$$\begin{pmatrix} -V_{1,h} & V_h & K_{1,h} \\ V_h & & -(\frac{1}{2}M_h + K_h) \\ K_{1,h}^\top & -(\frac{1}{2}M_h^\top + K_h^\top) & \varrho A_h + D_{1,h} \end{pmatrix} \begin{pmatrix} \underline{t} \\ \underline{q} \\ \underline{z} \end{pmatrix} = \begin{pmatrix} \underline{f}_1 \\ \underline{f}_2 \\ \underline{f}_3 \end{pmatrix}$$

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Schur complement system:

$$T_{\varrho,h}\underline{z} = \underline{f},$$

Lemma

$T_{\varrho,h}$ is positive definite, i.e.,

$$(T_{\varrho,h}\underline{z},\underline{z}) \geq \varrho(A_h\underline{z},\underline{z}) = \varrho\langle Az_h, z_h\rangle_\Gamma \geq \varrho\gamma_1^A \|z_h\|^2_{H^{1/2}(\Gamma)}$$

for all $\underline{z} \in \mathbb{R}^M \leftrightarrow z_h \in S_h^1(\Gamma)$.

Error Estimates

Theorem

Let $z \in H^{1/2}(\Gamma)$ *be the unique solution of the optimality system. Let* $z_h \in S_h^1(\Gamma)$ *be the unique solution of BEM system. Then there holds the error estimate*

$$\|z - z_h\|_{H^{1/2}(\Gamma)} \leq c(z, \overline{u}, f)\, h^{3/2}\,,$$

when assuming $u \in H^{5/2}(\Omega)$, $z \in H^2(\Gamma)$, *and* $t_z \in H^1_{pw}(\Gamma)$.

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$$\|z - z_h\|_{L_2(\Gamma)} \leq c(z, \overline{u}, f)\, h^2\,.$$

Remark:

- estimate hold for smooth Γ
- reduced order of convergence for Γ piecewise polygonal

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System

$$
\begin{aligned}
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-\Delta u &= f \quad \text{in } \Omega, \quad u = z \quad \text{on } \Gamma, \\
\frac{\partial}{\partial n_x} p &= \varrho A z \quad \text{on } \Gamma \quad .
\end{aligned}
$$

FE discretization:

$$
\begin{pmatrix} -M_{II} & K_{II} & -M_{CI} \\ K_{II} & & K_{CI} \\ M_{IC} & -K_{IC} & M_{CC} + \varrho A_h \end{pmatrix} \begin{pmatrix} \underline{u}_I \\ \underline{p} \\ \underline{z} \end{pmatrix} = \begin{pmatrix} -\underline{g}_I \\ \underline{f}_I \\ \underline{g}_C \end{pmatrix}.
$$

Finite Element Approximation

System

$$-\Delta p = u - \overline{u} \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \Gamma,$$
$$-\Delta u = f \quad \text{in } \Omega, \quad u = z \quad \text{on } \Gamma,$$
$$\frac{\partial}{\partial n_x} p = \varrho A z \quad \text{on } \Gamma \quad .$$

FE discretization:

$$\begin{pmatrix} -M_{II} & K_{II} & -M_{CI} \\ K_{II} & & K_{CI} \\ M_{IC} & -K_{IC} & M_{CC} + \varrho A_h \end{pmatrix} \begin{pmatrix} \underline{u}_I \\ \underline{p} \\ \underline{z} \end{pmatrix} = \begin{pmatrix} -\underline{g}_I \\ \underline{f}_I \\ \underline{g}_C \end{pmatrix}.$$

BE discretization:

$$\begin{pmatrix} -V_{1,h} & V_h & K_{1,h} \\ V_h & & -(\frac{1}{2} M_h + K_h) \\ K_{1,h}^\top & -(\frac{1}{2} M_h^\top + K_h^\top) & \varrho A_h + D_{1,h} \end{pmatrix} \begin{pmatrix} \underline{t} \\ \underline{q} \\ \underline{z} \end{pmatrix} = \begin{pmatrix} \underline{f}_1 \\ \underline{f}_2 \\ \underline{f}_3 \end{pmatrix}$$

Finite Element Approximation

FE discretization:

$$\begin{pmatrix} -M_{II} & K_{II} & -M_{CI} \\ K_{II} & & K_{CI} \\ M_{IC} & -K_{IC} & M_{CC} + \varrho A_h \end{pmatrix} \begin{pmatrix} \underline{u}_I \\ \underline{p} \\ \underline{z} \end{pmatrix} = \begin{pmatrix} -\underline{g}_I \\ \underline{f}_I \\ \underline{g}_C \end{pmatrix}.$$

Lemma

The Schur complement matrix

$$\widetilde{T}_{\varrho,h} = K_{IC} K_{II}^{-1} M_{II} K_{II}^{-1} K_{CI} - K_{IC} K_{II}^{-1} M_{CI} - M_{IC} K_{II}^{-1} K_{CI} + M_{CC} + \varrho A_h$$

is positive definite.

Finite Element Approximation

FE discretization:

$$\begin{pmatrix} -M_{II} & K_{II} & -M_{CI} \\ K_{II} & & K_{CI} \\ M_{IC} & -K_{IC} & M_{CC}+\varrho A_h \end{pmatrix} \begin{pmatrix} \underline{u}_I \\ \underline{p} \\ \underline{z} \end{pmatrix} = \begin{pmatrix} -\underline{g}_I \\ \underline{f}_I \\ \underline{g}_C \end{pmatrix}.$$

Lemma

The Schur complement matrix

$$\widetilde{T}_{\varrho,h} = K_{IC}K_{II}^{-1}M_{II}K_{II}^{-1}K_{CI} - K_{IC}K_{II}^{-1}M_{CI} - M_{IC}K_{II}^{-1}K_{CI} + M_{CC} + \varrho A_h$$

is positive definite.

Theorem

Error estimates of the FE approximations:

$$\|z - z_h\|_{H^{1/2}(\Gamma)} \le c(z,\overline{u},f)\,h \qquad \|u-u_h\|_{H^1(\Omega)} \le c(z,\bar{u},f)\,h$$

$$\|z - z_h\|_{L_2(\Gamma)} \le c(z,\overline{u},f)\,h^{3/2} \qquad \|u-u_h\|_{L_2(\Omega)} \le c(z,\bar{u},f)\,h^2$$

Outline

Dirichlet Boundary Control Problem

Boundary Integral Equations

Boundary Element Solution

Finite Element Approach

Numerical Results

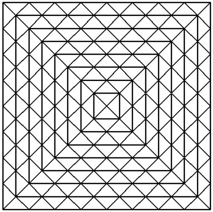
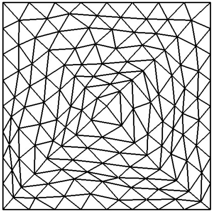
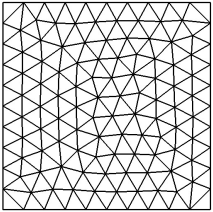
Numerical example: BEM and BEM

domain $\Omega = (0, \frac{1}{2})^2 \subset \mathbb{R}^2$, uniform mesh,

$$\overline{u}(x) = -\left(4 + \frac{1}{\varrho}\right)\left[x_1(1-2x_1) + x_2(1-2x_2)\right], \quad f(x) = -\frac{8}{\varrho}, \quad \varrho = 0.01\,.$$

	BEM		FEM		FEM, L_2	
L	$\|\|z_L - z_9^B\|\|$	eoc	$\|\|z_L - z_9^F\|\|$	eoc	$\|\|z_L - z\|\|$	eoc
2	2.25e-0		3.89e-1		4.50e-01	
3	4.66e-1	2.27	1.07e-1	1.86	1.77e-01	1.35
4	8.84e-2	2.39	2.81e-2	1.93	7.94e-02	1.15
5	1.63e-2	2.44	7.28e-3	1.95	3.38e-02	1.23
6	3.02e-3	2.43	1.87e-3	1.96	1.34e-02	1.34
7	5.73e-4	2.40	4.69e-4	2.00	5.02e-03	1.41
8	1.24e-4	2.20	1.06e-4	2.15	1.83e-03	1.45

Numerical example: FEM several meshes

L	N	$\|z_L - z_9^F\|$	eoc	$\|z_L - z_9^F\|$	eoc	N	$\|z_L - z_9^F\|$	eoc
3	256	1.07e-01		1.25e-01		216	1.65e-01	
4	1024	2.81e-02	1.94	3.26e-02	1.94	864	4.69e-02	1.81
5	4096	7.28e-03	1.95	8.45e-03	1.95	3456	1.29e-02	1.87
6	16384	1.87e-03	1.96	2.17e-03	1.96	13824	3.39e-03	1.93
7	65536	4.69e-04	2.00	5.43e-04	2.00	55296	8.41e-04	2.01
8	262144	1.06e-04	2.15	1.20e-04	2.18	221184	1.76e-04	2.26

Numerical example: $\overline{u}$ singular

domain $\Omega = (0, \frac{1}{2})^2 \subset \mathbb{R}^2$, uniform mesh,

$$\overline{u} = (x_1^2 + x_2^2)^{-1/3}, \quad f = 0, \quad , \varrho = 0.01$$

L	N	BEM $\|\|z_L - z_9^B\|\|$	BEM eoc	FEM $\|\|z_L - z_9^B\|\|$	FEM eoc	FEM216 N	FEM216 $\|\|z_L - z_9^F\|\|$	FEM216 eoc
3	256	7.00e-2		3.61e-02		216	9.74e-02	
4	1024	2.13e-2	1.71	1.55e-02	1.22	864	4.10e-02	1.25
5	4096	6.46e-3	1.72	5.88e-03	1.40	3456	1.52e-02	1.43
6	16384	1.94e-3	1.74	2.08e-03	1.50	13824	5.18e-03	1.55
7	65536	5.67e-4	1.77	6.88e-04	1.60	55296	1.62e-03	1.67
8	262144	1.54e-4	1.88	1.91e-04	1.85	221184	4.19e-04	1.95

Outlook

- partial Dirichlet boundary control
- BEM
- box constraints
- time dependent problems
- fast solvers