Optimal control of the Stokes equations: The control constrained case under reduced regularity

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Model problem

We discuss the optimal control problem

$$J(\bar{u}) = \min J(u),$$
$$J(u) := F(Su, u),$$
$$F(v, u) := \frac{1}{2}\|v - v_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)},$$

where the associated velocity $v = Su$ to the control u is the weak solution of the state equation

$$\begin{aligned} -\Delta v + \nabla p &= u \quad \text{in } \Omega \\ \nabla \cdot v &= 0 \quad \text{in } \Omega \\ v &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

and the control variable is constrained by

$$a \le u(x) \le b \qquad \text{for a.a. } x \in \Omega.$$

$\Omega \subset \mathbb{R}^d$ is a domain with corners and/or edges.

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Results from literature

control-constrained, scalar elliptic state equation:

- Hinze, 2004:

  variational-discrete approach, full regularity $\rightarrow$ second order convergence

- Meyer, Rösch 2004:

  post-processing approach, full regularity $\rightarrow$ second order convergence

- Apel, Rösch, Winkler 2005 and Apel, Winkler 2007:

  post-processing and variational discrete approach, edge and/or corner singularities, isotropic mesh-grading $\rightarrow$ second order convergence

- Apel, S., Winkler 2008:

  post-processing and variational discrete approach, edge singularities, anisotropic mesh-grading $\rightarrow$ second order convergence

control-constrained, Stokes equation:

- Rösch, Vexler 2006:

  post-processing approach, velocity field in $H^2(\Omega) \cap W^{1,\infty}(\Omega)$ $\rightarrow$ second order convergence

Weak formulation of the state equation

We introduce the spaces

$$X = \left\{ v \in (H^1(\Omega))^d : v|_{\partial\Omega} = 0 \right\},$$
$$M = \left\{ p \in L^2(\Omega) : \int_\Omega p = 0 \right\}$$

and the bilinear forms $a : X \times X \to \mathbb{R}$ and $b : X \times M \to \mathbb{R}$ as

$$a(v, \varphi) := \sum_{i=1}^{d} \int_\Omega \nabla v_i \cdot \nabla \varphi_i \quad \text{and} \quad b(\varphi, p) := -\int_\Omega p \nabla \cdot \varphi.$$

The weak solution $(v, p) \in X \times M$ of the state equation is given as unique solution of

$$\begin{aligned} a(v, \varphi) + b(\varphi, p) &= (u, \varphi) \qquad && \forall \varphi \in X \\ b(v, \psi) &= 0 && \forall \psi \in M. \end{aligned}$$

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Regularity

Weighted Sobolev spaces $H^k_\omega(\Omega)^d$, $k = 1,2$ with norm

$$\|y\|_{H^k_\omega(\Omega)^d} := \left(\sum_{|\alpha|\le k} \|\omega_\alpha D^\alpha y\|^2_{L^2(T)^d} \right)^{1/2}$$

where ω_α is a suitable positive weight depending on the concrete problem under consideration.

Assumption:

1. The a priori estimate

$$\|v\|_{H^2_\omega(\Omega)^d} \le c\|u\|_{L^2(\Omega)^d}$$

is valid and the embedding

$$H^2_\omega(\Omega) \hookrightarrow C^{0,\sigma}(\Omega), \quad \sigma \in (0,1)$$

holds.

Discrete spaces

The approximation of

- the control u is contained in U_h, the space of piecewise constant functions,

$$U_h = \left\{ u_h \in U : u_h|_T \in (\mathcal{P}_0)^d \text{ for all } T \in \mathcal{T}_h \right\},$$

- the pressure p is contained in a space of piecewise polynomial functions $M_h \subset M$
- the velocity v is contained in a space of piecewise polynomial functions $X_h \subset X$ or $X_h \not\subset X$

Discretized state equation

We define the weaker bilinear forms $a_h : X_h \times X_h \to \mathbb{R}$ and $b_h : X_h \times M_h \to \mathbb{R}$ with

$$a_h(v_h, \varphi_h) := \sum_{T \in \mathcal{T}_h} \sum_{i=1}^{d} \int_T \nabla v_{h,i} \cdot \nabla \varphi_{h,i}$$

$$b_h(\varphi_h, p_h) := - \sum_{T \in \mathcal{T}_h} \int_T p_h \nabla \cdot \varphi_h.$$

For a given control $u_h \in U_h$ the discretized state equation reads as

Find $(v_h, p_h) \in X_h \times M_h$ such that

$$\begin{aligned} a_h(v_h, \varphi_h) + b_h(\varphi_h, p_h) &= (u_h, \varphi_h) \qquad && \forall \varphi_h \in X_h \\ b_h(v_h, \psi_h) &= 0 && \forall \psi_h \in M_h. \end{aligned}$$

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Discretized optimality system

State equation:

$$
\begin{aligned}
a_h(\bar{v}_h, \varphi_h) + b_h(\varphi_h, \bar{p}_h) &= (\bar{u}_h, \varphi_h) \qquad && \forall \varphi_h \in X_h \\
b_h(v_h, \psi_h) &= 0 && \forall \psi_h \in M_h
\end{aligned}
$$

Adjoint equation:

$$
\begin{aligned}
a_h(\bar{w}_h, \varphi_h) - b_h(\varphi_h, \bar{r}_h) &= (\bar{v}_h - v_{d,h}, \varphi_h) \qquad && \forall \varphi_h \in X_h \\
b_h(\bar{w}_h, \psi_h) &= 0 && \forall \psi_h \in M_h
\end{aligned}
$$

Variational inequality:

$$
(\bar{w}_h + \nu \bar{u}_h, u_h - \bar{u}_h)_{L^2(\Omega)} \geq 0 \quad \forall u_h \in U_h^{ad}
$$

Assumptions on the space X_h

2. The discrete Poincaré inequality

$$\|v_h\|_{L^2(\Omega)} \leq c\|v_h\|_{X_h} \qquad \forall v_h \in X_h$$

holds where $\|\cdot\|_{X_h} = a_h(\cdot,\cdot)^{1/2}$.

3. There exists a $p \leq \frac{2d}{d-2}$, such that the inverse estimate

$$\|\varphi_h\|_{L^\infty(\Omega)} \leq ch^{-1}\|\varphi_h\|_{L^p(\Omega)} \qquad \forall \varphi_h \in X_h$$

is valid.

Assumptions on the spaces X_h and M_h (1)

4. The consistency error estimate

$$|a_h(v,\varphi_h) + b_h(\varphi_h, p) - (u,\varphi_h)| \le ch\|\varphi_h\|_{X_h}\|u\|_{L^2(\Omega)} \qquad \forall (u,\varphi_h) \in L^2(\Omega) \times X_h$$

holds.

5. The pair (X_h, M_h) fulfills the uniform discrete inf-sup-condition, i.e. there exists a positive constant β independent of h such that

$$\inf_{\psi_h \in M_h} \sup_{\varphi_h \in X_h} \frac{b(\varphi_h,\psi_h)}{\|\varphi_h\|_{X_h}\|\psi\|_M} \ge \beta.$$

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Assumptions on the spaces X_h and M_h (2)

6. There exist interpolation operators

$$i_h^v : H^2_\omega(\Omega)^d \cap X \to X_h \cap X$$

and

$$i_h^p : H^1_\omega(\Omega) \cap M \to M_h$$

such that for the solution $(v, p) \in X \times M$ of the state equation the interpolation properties

(i) $\|v - i_h^v v\|_{X_h} \le ch\|v\|_{H^2_\omega(\Omega)} \le ch\|u\|_{L^2(\Omega)}$
(ii) $\|v - i_h^v v\|_{L^\infty(\Omega)} \le c\|u\|_{L^2(\Omega)}$
(iii) $\|p - i_h^p p\|_{L^2(\Omega)} \le ch\|p\|_{H^1_\omega(\Omega)} \le ch\|u\|_{L^2(\Omega)}$.

hold.

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Projection operators Q_h and R_h

The operator R_h projects continuous functions in the space of piecewise constant functions,

$$(R_h f)(x) := f(S_T) \ \text{ if } x \in T$$

where S_T denotes the centroid of the element T.
The operator Q_h projects L^2-functions in the space of piecewise constant functions,

$$(Q_h g)(x) := \frac{1}{|T|} \int_T g(x)\,\mathrm{d}x \ \text{ for } x \in T.$$

Assumptions on the operators Q_h and R_h

7. The optimal control $\bar{u}$ and the corresponding adjoint state $(\bar{w}, \bar{r})$ satisfy the inequality

$$\|Q_h \bar{w} - R_h \bar{w}\|_{L^2(\Omega)} \leq ch^2 \left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})} \right).$$

8. For the optimal control $\bar{u}$ and all functions $\varphi_h \in X_h$ the inequality

$$(Q_h \bar{u} - R_h \bar{u}, \varphi_h)_{L^2(\Omega)} \leq ch^2 \|\varphi_h\|_{L^\infty(\Omega)} \left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})} \right)$$

holds.

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Summary of assumptions

1. Regularity in $H^2_\omega(\Omega)^d$
2. discrete Poincaré inequality
3. inverse estimates
4. consistency error estimate
5. uniform discrete inf-sup-condition
6. interpolation error estimates
7. Estimate for $\|Q_h\bar{w} - R_h\bar{w}\|_{L^2(\Omega)}$
8. Estimate for $(Q_h\bar{u} - R_h\bar{u}, \varphi_h)_{L^2(\Omega)}$

Supercloseness result

The following supercloseness result [Meyer/Rösch 2004] extends to our setting.

Theorem

The inequality

$$\|\bar{u}_h - R_h\bar{u}\|_U \leq ch^2 \left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^d} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})^d} \right)$$

is valid.

The proof relies on the assumption 7,

$$\|Q_h\bar{w} - R_h\bar{w}\|_{L^2(\Omega)} \leq ch^2 \left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})} \right)$$

Superconvergence result

Post-processing step:

$$\tilde{u}_h = \Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}\bar{w}_h\right).$$

Theorem

Assume that the assumptions 1–8 hold. Then the estimates

$$\|\bar{v} - \bar{v}_h\|_U \le ch^2\left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^d} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})^d}\right),$$
$$\|\bar{w} - \bar{w}_h\|_U \le ch^2\left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^d} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})^d}\right),$$
$$\|\bar{u} - \tilde{u}_h\|_U \le ch^2\left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^d} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})^d}\right)$$

are valid with a positive constant c independent of h.

Superconvergence result

Post-processing step:

$$\tilde{u}_h = \Pi_{[u_a,u_b]}\left(-\frac{1}{\nu}\bar{w}_h\right).$$

Theorem

Assume that the assumptions 1–8 hold. Then the estimates

$$\|\bar{v}-\bar{v}_h\|_U \le ch^2\left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^d} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})^d}\right),$$

$$\|\bar{w}-\bar{w}_h\|_U \le ch^2\left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^d} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})^d}\right),$$

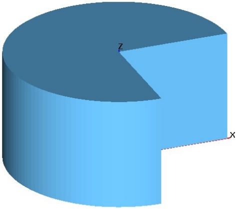
$$\|\bar{u}-\tilde{u}_h\|_U \le ch^2\left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^d} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})^d}\right)$$

are valid with a positive constant c independent of h.

Idea of the proof:

$$\|\bar{v}-\bar{v}_h\|_U = \|S\bar{u}-S_h\bar{u}_h\|_U$$
$$\le \|S\bar{u}-S_h\bar{u}\|_U + \|S_h\bar{u}-S_hR_h\bar{u}\|_U + \|S_hR_h\bar{u}-S_h\bar{u}_h\|_U.$$

Example: 3D domain, anisotropic mesh grading (1)



$\Omega = G \times Z$, where $G \subset \mathbb{R}^2$ is a bounded polygonal domain with one reentrant corner with interior angle ω and $Z := (0, z_0) \subset \mathbb{R}$ is an interval.

For $\lambda \in \mathbb{R}$ being the smallest positive solution of

$$\sin(\lambda\omega) = -\lambda \sin\omega.$$

one has ([Maz'ya,Plamenevskiĭ,83], [Apel,Nicaise,Schöberl,01])

$$\|v\|_{V_\beta^{2,p}(\Omega)^3} + \|p\|_{V_\beta^{1,p}(\Omega)} \leq c\|u\|_{L^p(\Omega)}, \ \beta > 1 - \lambda$$

$$\|\partial_3 v\|_{V_0^{1,2}(\Omega)^3} + \|\partial_3 p\|_{L^2(\Omega)} \leq c\|u\|_{L^2(\Omega)}.$$

where

$$\|v\|_{V_\beta^{k,p}(\Omega)} := \left(\int_\Omega \sum_{|\alpha|\leq k} r^{p(\beta-k+|\alpha|)} |D^\alpha v|^p \mathrm{d}x\right)^{1/p}.$$

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Example: 3D domain, anisotropic mesh grading (2)

Let $v_d \in C^{0,\sigma}(\bar{\Omega})$, $\sigma \in (0, 1/2)$ and $\gamma > 1 - \lambda$. Then the inequality

$$\|r^\gamma \nabla P\bar{u}\|_{L^\infty(\Omega)^3} \leq c \left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})^3} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})^3} \right)$$

is valid.

Remarks:

- for non-convex domains ($\omega > \pi$): $\frac{1}{2} < \lambda < \frac{\pi}{\omega}$, d.h. $\gamma > 0$.
- for convex domains ($\omega < \pi$): $\lambda > 1 \Rightarrow W^{1,\infty}$-regularity of the solution.
- One can weaken the condition $\omega < \frac{2}{3}\pi$ in [Rösch,Vexler '06] to $\omega < \pi$ to guarantee $W^{1,\infty}$-regularity of the velocity field.

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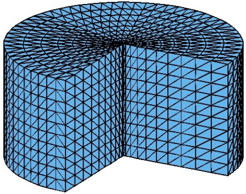
Example: 3D domain, anisotropic mesh grading (3)

- anisotropic, graded mesh, grading parameter $\mu < \lambda$



$$h_{T,i} \sim h^{1/\mu} \quad \text{for } r_T = 0,$$
$$h_{T,i} \sim h r_T^{1-\mu} \quad \text{for } r_T > 0,$$
$$h_{T,3} \sim h,$$

for $i = 1, 2$ and $r_T := \inf_{(x_1,x_2)\in T}(x_1^2 + x_2^2)^{1/2}$.

- Velocity: Crouzeix-Raviart finite element space

$$X_h := \left\{ v_h \in L^2(\Omega)^3 : v_h|_T \in (\mathcal{P}_1)^3 \ \forall T, \int_F [v_h]_F = 0 \ \forall F \right\},$$

i.e. $X_h \not\subset X$.

- Pressure: piecewise constant functions,

$$M_h := \left\{ q_h \in L^2(\Omega) : q_h|_T \in \mathcal{P}_0 \ \forall T, \int_\Omega q_h = 0 \right\}.$$

Example: 3D domain, anisotropic mesh grading (4)

Theorem

In the described setting the assumptions 1–8 are fullfilled.
Control, state and adjoint state converge with second order.

Remarks on the proof:

- Discrete Poincaré inequality is proved in [Lazaar,Nicaise 2002].
- Interpolation error estimates: use quasi-interpolant [Apel,S, 2008]
- consistency error estimate and discrete inf-sup-condition are proved in [Apel, Nicaise, Schöberl, 2001]
- For the proof of

$$\|Q_h\bar{w} - R_h\bar{w}\|_{L^2(\Omega)} \leq ch^2 \left(\|\bar{u}\|_{C^{0,\sigma}(\bar{\Omega})} + \|v_d\|_{C^{0,\sigma}(\bar{\Omega})}\right)$$

  split Ω in two parts, namely one containing the elements along the edge, the other one the elements away from the edge and use “anisotropic” regularity and $r^\gamma \nabla \bar{w} \in L^\infty(\Omega)^3$ for $\gamma > 1 - \lambda$.

Numerical test: 3D domain, anisotropic mesh grading

Modified functional:

$$J(v,u) := \frac{1}{2}\|v - v_d\|^2_{L^2(\Omega)^d} + \frac{\nu}{2}\|u\|_{L^2(\Omega)^d} + \int_{\partial\Omega} \frac{\partial v}{\partial n} g \, \mathrm{ds}$$

The functions f, g and v_d are chosen such that

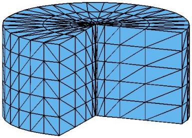
$$\bar{v} = \bar{w} = \begin{bmatrix} x_3 r^\lambda \Phi_1(\varphi) \\ x_3 r^\lambda \Phi_2(\varphi) \\ r^{2/3} \sin \frac{2}{3}\varphi \end{bmatrix}, \quad \bar{p} = -\bar{r} = x_3 r^{\lambda-1}\Phi_p(\varphi), \quad \bar{u} = \Pi_{[-10,-0.2]}\left(-\frac{1}{\nu}\bar{w}\right)$$

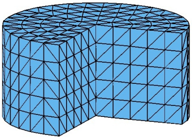
with

$$\begin{aligned} \Phi_1(\varphi) &= -\sin(\lambda\varphi)\cos\omega - \lambda\sin(\varphi)\cos(\lambda(\omega-\varphi)+\varphi) \\ &\quad + \lambda\sin(\omega-\phi)\cos(\lambda\varphi-\varphi) + \sin(\lambda(\omega-\varphi)), \\ \Phi_2(\varphi) &= -\sin(\lambda\varphi)\sin\omega - \lambda\sin(\varphi)\sin(\lambda(\omega-\varphi)+\varphi) \\ &\quad - \lambda\sin(\omega-\varphi)\sin(\lambda\varphi-\varphi), \\ \Phi_p(\varphi) &= 2\lambda\left[\sin((\lambda-1)\varphi+\omega) + \sin((\lambda-1)\varphi-\lambda\omega)\right]. \end{aligned}$$

is the exact solution for the optimal control problem.

Numerical test: 3D domain, anisotropic mesh grading





	$\mu = 0.4 < \lambda \approx 0.5445$		$\mu = 1$	
ndof	value	rate	value	rate
14025	1.11E-02		1.38E-02	
37779	6.07E-03	1.83	8.86E-03	1.33
108600	3.19E-03	1.83	5.66E-03	1.27
362475	1.49E-03	1.89	3.46E-03	1.23
854400	8.64E-04	1.91	2.45E-03	1.20
1135464	7.20E-04	1.93	2.20E-03	1.18
1663125	5.62E-04	1.94	1.89E-03	1.17

Table: L^2-error of the computed control $\tilde{u}_h$ on an anisotropic, three-dimensional mesh

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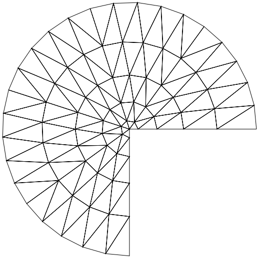
Example: 2D non-convex domain, isotropic mesh grading

- isotropic, graded mesh, grading parameter $\mu < \lambda$

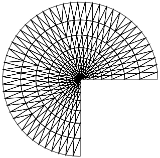


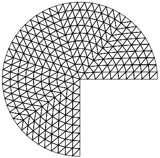
$$h_\tau \sim \begin{cases} h^{1/\mu} & \text{for } r_\tau = 0, \\ h r_\tau^{1-\mu} & \text{for } 0 < r_\tau \le R, \\ h & \text{for } r_\tau > R. \end{cases}$$

with $r_T := \inf_{(x_1,x_2)\in T}(x_1^2 + x_2^2)^{1/2}$.

- different element pairs
  - $(\mathcal{P}_2^c, \mathcal{P}_0)$
  - Taylor-Hood element
  - Bernardi-Raugel-Fortin element
  - Mini-element
  - ...

Numerical test: 2D non-convex domain, isotropic mesh grading





	$\mu = 0.4 < \lambda \approx 0.5445$		$\mu = 1$	
ndof	value	rate	value	rate
10282	8.02E-04	1.94	1.16E-03	1.52
40562	2.08E-04	1.96	4.16E-04	1.49
161122	5.32E-05	1.98	1.52E-04	1.46
392377	2.20E-05	1.99	8.08E-05	1.42
642242	1.35E-05	1.99	5.73E-05	1.39
2564482	3.38E-06	1.99	2.25E-05	1.35

Table: L^2-error of the computed control $\tilde{u}_h$ on an isotropic, two-dimensional mesh with $(\mathcal{P}_2^c, \mathcal{P}_0)$

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