Optimal control of variational inequalities

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Outline

1. **Introduction**
2. Optimality conditions
3. Regularization and path analysis

Minimize

$$J(y,u) := g(y) + j(u)$$

subject to the variational inequality $y \in K$

$$a(y, v - y) \geq (u, v - y) \quad \forall v \in K$$

with

$$K = \{y \in H_0^1(\Omega) : \ y \leq \psi\}.$$

References: Barbu, Bergounioux, Bonnans, Casas, Hintermüller, Ito, Kunisch, Mordukhovich, Mignot, Outrata, Puel, Tiba, . . .

Bilinear form: $a(\cdot,\cdot)$ coercive, induced by 2^{nd}-order differential operator A with smooth coefficients

Obstacle: $\psi \in H^1(\Omega)$, $\psi|_\Gamma \geq 0$, $A\psi \in L^2(\Omega)$

Unique solvability: For every $u \in L^2(\Omega)$, the v.i. admits a unique solution $y \in H_0^1(\Omega) \cap H^2$

Multiplier: $\lambda := u - Ay$ with $\lambda \geq 0$, $\langle \lambda, y - \psi \rangle = 0$

Control-to-state mapping:

$u \mapsto (y, \lambda)$ is directional differentiable as $H^{-1}(\Omega) \to H_0^1(\Omega) \times H^{-1}(\Omega)$.

But neither differentiable nor Lipschitz as $L^2(\Omega) \to H^2(\Omega) \times L^2(\Omega)$, hence pointwise discussion w.r.t. λ is difficult.

Optimal control problem equivalent to

$$\begin{aligned} &\min J(y(u), u) \\ &\text{s.t. } \psi - y(u) \geq 0,\ \lambda(u) \geq 0,\ \langle y - \psi, \lambda \rangle = 0 \end{aligned}$$

Mathematical programming with complementarity constraints:

$$\min f(z) \text{ s.t. } F(x) \geq 0,\ G(z) \geq 0,\ F(x)^T G(x) = 0.$$

Standard constraint qualifications fail, extensive literature on specialized MPCC-CQ.

$\Rightarrow$ MPCC-CQ are satisfied (formally) for our problem

Bi-level formulation

The variational inequality

$$a(y, v - y) \geq (u, v - y) \quad \forall v \in K$$

is the necessary and sufficient optimality condition of a constrained minimization problem.

Bi-level optimization problem:

$$\min_{u \in L^2(\Omega),\ y \in H_0^1(\Omega)} J(y, u)$$

$$y = \operatorname*{argmin}_{y \in K} \frac{1}{2} a(y, y) - (u, y)$$

Consider the state constraint problem:

$$\min J(y, u)$$

subject to

$$Ay = u, \quad y \leq \psi.$$

Equivalent bi-level formulation:

$$\min_{u \in L^2(\Omega),\; y \in K} J(y, u)$$

$$y = \operatorname*{argmin}_{y \in H_0^1(\Omega)} \frac{1}{2} a(y, y) - (u, y)$$

Control of variational inequality: **(non-convex)**

$$\min_{u\in L^2(\Omega),\ y\in H_0^1(\Omega)} J(y,u)$$

$$y = \operatorname*{argmin}_{y\in K} \frac{1}{2}a(y,y) - (u,y)$$

State constrained problem: **(convex)**

$$\min_{u\in L^2(\Omega),\ y\in K} J(y,u)$$

$$y = \operatorname*{argmin}_{y\in H_0^1(\Omega)} \frac{1}{2}a(y,y) - (u,y)$$

In the control problem for the variational inequality

- the state constraint is imposed in the inner optimization problem
- every control is feasible (i.e. gives a feasible state)

1. Introduction

2. **Optimality conditions**

3. Regularization and path analysis

Let $\bar{u} \in L^2(\Omega)$ with $\bar{y} \in H_0^1(\Omega) \cap H^2(\Omega)$, $\bar{\lambda} \in L^2(\Omega)$ be locally optimal.

Theorem: [Mignot '76, Mignot-Puel '84]

There exist uniquely determined adjoint states $\bar{p} \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and $\bar{\mu} \in H^{-1}(\Omega) \cap (L^\infty(\Omega))^*$ fulfilling:

- **Adjoint equation and optimality condition**

$$A^*\bar{p} + \bar{\mu} + g'(\bar{y}) = 0, \qquad j'(\bar{u}) - \bar{p} = 0$$

- **Complementarity conditions**

$$\bar{\lambda}\bar{p} = 0 \text{ a.e. on } \Omega,$$

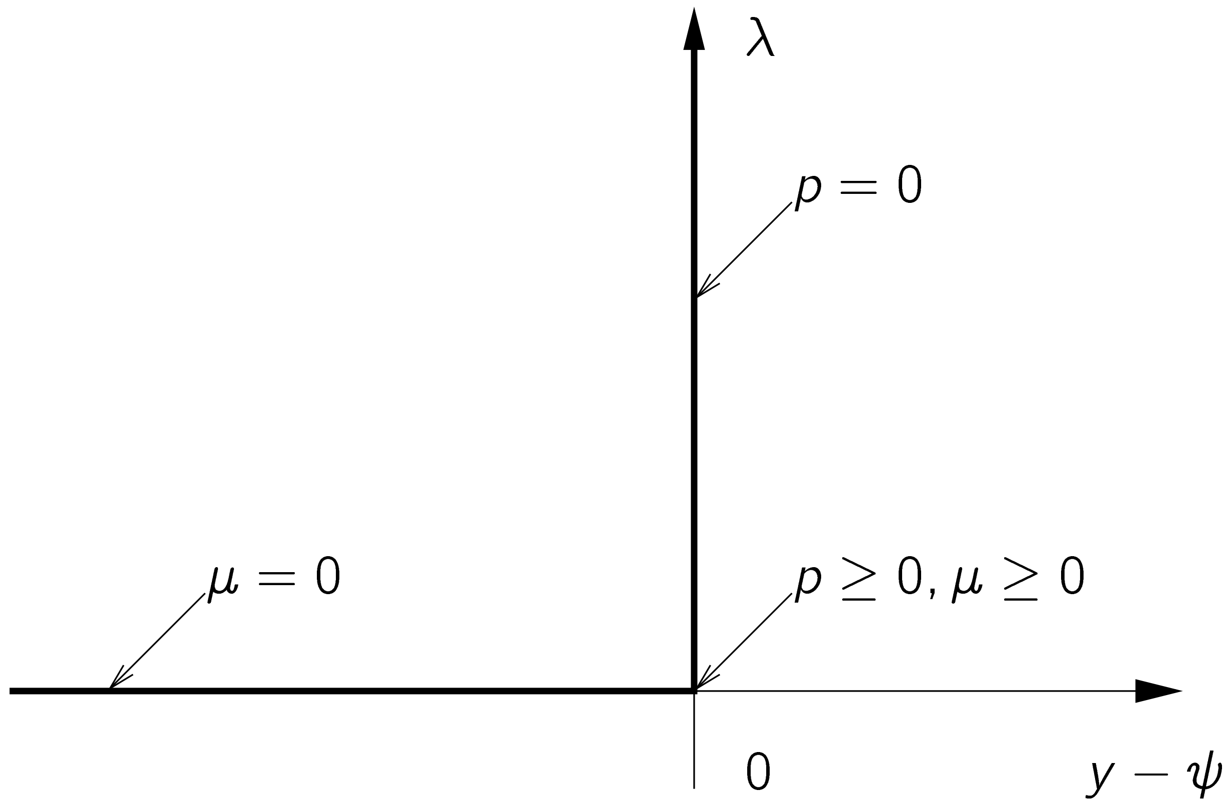
$$\langle \bar{\mu}, \varphi(\bar{y} - \psi) \rangle = 0 \text{ for all } \varphi \in C^1(\bar{\Omega}) \text{ such that } \varphi\psi|_\Gamma = 0,$$

- **Sign conditions**

$\bar{p} \geq 0$ where $\bar{y} = \psi$, $\langle \bar{\mu}, \bar{p} \rangle \geq 0$,

$\langle \bar{\mu}, \phi \rangle \geq 0$ for all $\phi \in H_0^1(\Omega)$ with $\langle \bar{\lambda}, \phi \rangle = 0$ and $\phi \geq 0$ on $\{\bar{y} = \psi\}$.

Optimality system similar to **strong stationarity** in mpcc's.



Open: prove strong stationarity for problems with additional control- and state constraints [Hintermüller, Kopacka '08] [Outrata, Jarusek, Stara '09]

Lagrange formalism

Lagrange-function:

$$L = J(y,u) + \langle Ay + \lambda - u, p\rangle - \langle p, \lambda\rangle + \langle \mu, y - \psi\rangle$$

- p multiplier to $\lambda \geq 0$
- μ multiplier to $y \leq \psi$
- in general no multiplier exists for $\lambda(y - \psi) = 0$
  [Bergounioux, Mignot '00]

Comparison to state-constrained problems:

- Only information about signs of $\bar{p}$ and $\bar{\mu}$ on part of the domain!
- No interior point assumption is needed.

Sufficient optimality condition

1. There is $\gamma > 0$ such that

$$j''(\bar{u})(h,h) \geq \gamma \|h\|_{L^2}^2 \quad \text{for all } h \in L^2(\Omega).$$

2. For all $h \in L^2(\Omega) \setminus \{0\}$ and $z = y'(\bar{u}; h)$ with $j'(\bar{u})h + g'(\bar{y})z = 0$, we have

$$g''(\bar{y})(z,z) + j''(\bar{u})(h,h) > 0.$$

3. There exists a constant $\tau > 0$ such that

$$\bar{p} \geq 0 \text{ on } \{\psi - \tau < \bar{y} < \psi\}.$$

4. Moreover, $\bar{\mu}$ satisfies

$$\langle \bar{\mu}, \phi \rangle \geq 0, \quad \phi \in H_0^1(\Omega), \phi \geq 0.$$

Sufficient optimality condition

Theorem: [Kunisch, W '09]

Let $(\bar{y}, \bar{u}, \bar{\lambda}, \bar{p}, \bar{\mu})$ fulfill the optimality system. If assumptions (1)–(4) are satisfied then $\bar{u}$ is locally optimal and it holds

$$J(y,u) \geq J(\bar{y},\bar{u}) + \alpha \|u - \bar{u}\|_{L^2}^2 \quad \text{for all } u \in L^2(\Omega),\ \|u - \bar{u}\|_{L^2} \leq \rho,$$

with some $\alpha, \rho > 0$.

Open: Stability of the sufficient condition.

Comparison to finite-dimensional mpcc:

Local decomposition approach not applicable: Small changes in $\bar{u}, \bar{y}, \bar{\lambda}$ can cause changes of the active / inactive sets.

$\Rightarrow$ need stronger sign conditions on $\bar{p}$ and $\bar{\mu}$ than obtained by strong stationarity.

Comparison to state constrained problems:

$\bar{p}$ and $\bar{\mu}$ can be regarded as multipliers to $\lambda \geq 0$ and $y \leq \psi$, but:

- incomplete information about signs of $\bar{p}$ and $\bar{\mu}$
- the mapping $u \mapsto \lambda$ is not continuous from $L^2(\Omega)$ to $L^\infty(\Omega)$.

We can weaken the assumption if we want to prove local optimality with respect to the norm $\|u\|_{L^2} + \|\lambda\|_{L^\infty}$.

1. Introduction
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Smoothing the state equation

Starting point: The multiplier λ fulfills

$$\lambda = \max(0,\, \lambda + c(y-\psi)) \quad \forall c > 0.$$

Smoothing:

$$\lambda_c = f_c(\tilde\lambda + c(y-\psi)) \quad c > 0, \tilde\lambda > 0$$

Regularized equation:

$$Ay + f_c(\tilde\lambda + c(y-\psi)) = u$$

Feasibility: If $\tilde\lambda > \Lambda$ for some large $\Lambda > 0$ then y_c is feasible, $y_c \le \psi$.

Convergence for $c \to \infty$:

$$u_c \to u \text{ in } H^{-1}(\Omega) \Rightarrow \begin{cases} y_c(u_c) \to y(u) \text{ in } H_0^1(\Omega) \\ \lambda_c(u_c) \to \lambda(u) \text{ in } H^{-1}(\Omega) \end{cases}$$

Minimize $J(y,u)$ subject to

$$Ay + f_c(\tilde{\lambda} + c(y-\psi)) = u.$$

$\rightarrow$ no inequality constraints

Convergence: Global solutions (y_c, u_c) converge to global solutions of the original problem. [Ito, Kunisch '00]

Multipliers:

$$A^* p_c + f_c'(\tilde{\lambda} + c(y-\psi))p_c + g'(y_c) = 0, \quad j'(u_c) - p_c = 0.$$

Convergence of multipliers:

$$f_c(\dots) =: \lambda_c \rightarrow \bar{\lambda} \text{ in } H^{-1}(\Omega)$$

$$p_c \rightharpoonup \bar{p} \text{ in } H_0^1(\Omega)$$

$$f_c'(\dots)p_c =: \mu_c \rightharpoonup \bar{\mu} \text{ in } H^{-1}(\Omega)$$

Assumption: $j(u) = \frac{\gamma}{2}\|u\|_{L^2}^2$

Existence: For each strict local minimizer $(\bar{y}, \bar{u})$ there exists a family $(y_c, u_c)_{c>0}$ of local solutions of the regularized problem.

Convergence:

$$(y_c, u_c, \lambda_c) \to (\bar{y}, \bar{u}, \bar{\lambda}) \qquad \text{in } H_0^1(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega)$$

$$(p_c, \mu_c) \rightharpoonup (\bar{p}, \bar{\mu}) \qquad \text{in } H_0^1(\Omega) \times H^{-1}(\Omega)$$

Question: Is the path $c \mapsto u_c$ continuous? Or even Lipschitz or differentiable?

Value function: The function $V(c) := J(y_c, u_c)$ is continuous.

Path continuity: Under a modified second-order condition on $(\bar{y}, \bar{u})$ and some positivity assumptions on p_c $^{(*)}$, it holds:

> **Theorem:** [Kunisch, W '09]
> The mapping $c \mapsto u_c$ has a finite number of discontinuities. Hence, there is C_1 such that $c \mapsto u_c$ is continuous for all $c > C_1$ with respect to the strong topology of $L^2(\Omega)$.

Differentiability: If the path $c \mapsto (y_c, u_c, p_c)$ is continuous at c_0 then it is also **locally Lipschitz continuous** at c_0. Moreover, the path is **Gateaux differentiable** at c_0 if it is continuous in a neighborhood of c_0.

$^{(*)}$ Due to only weak convergence $p_c \rightharpoonup \bar{p}$ in $H^{-1}(\Omega)$, we cannot use $\bar{p} \geq \tau > 0$ to prove $p_c \geq 0$.

Outlook

Further work:

- Investigate properties of the value function (monotonicity, convexity / concavity),
- Study path-following strategies for $c \to \infty$.

Thank you very much!